Commutative Algebra and Algebraic Geometry in Milan

Politecnico di Milano, June 29–July 3, 2026

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Adequate complete intersection homomorphisms

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Introduction

The definition of complete intersection homomorphisms progressively became broader throughout the years since Marot's definition in the 80's. From a certain point of view, such a class of homomorphisms should satisfy the following properties.

We define a class $\mathcal{C}$ of local homomorphism of noetherian local rings to be *ci-adequate* if for any $f\colon (A,\mathfrak{m},k)\to(B,\mathfrak{n},l)$ in $\mathcal{C}$, the following statements hold:

(A1) Any two of the following properties imply the remaining one:
- (a) A is a complete intersection ring.
- (b) B is a complete intersection ring.
- (c) $f\in\mathcal{C}$.

(A2) If $f\in\mathcal{C}$, A is Gorenstein if and only if B is Gorenstein.

(A3) If $f\in\mathcal{C}$, A is Cohen-Macaulay if and only if B is Cohen-Macaulay.

Our objective is to find a "natural" class of local homomorphisms which is adequate in this sense, because the main classes of homomorphisms known to date in this context present the following problems:

For Avramov's definition of complete intersection (*ci*) homomorphisms [2] (those for which André-Quillen homology $\mathrm{H}_n(A,B,l)=0$ for $n\geq 2$), statement (i) fails: consider $(A,\mathfrak{m},k)\to k$ for any complete intersection which is not regular. Marot's definition is even more restrictive, so (i) fails as well.

For quasi-complete intersection (*qci*) homomorphisms [3] ($\mathrm{H}_n(A,B,l)=0$ for $n\geq 3$), whether B CM and $A\to B$ *qci* implies A CM is unknown.

New classes [1]

Feebly complete intersection (*fci*) $f\colon (A,\mathfrak{m},k)\to(B,\mathfrak{n},l)$
There exists a flat local homomorphism of noetherian local rings $A\to A'$ and a local homomorphism of noetherian local rings $Q\to A'$ satisfying

$$\mathrm{H}_n(Q,A',\ell')=0=\mathrm{H}_n(Q,B\otimes_A A',\ell')$$

for all $n\geq 2$, where $l'=(B\otimes_A A')/\mathfrak{n}'$ for $\mathfrak{n}'$ a certain maximal ideal.

Moderately complete intersection (*mci*) $f\colon (A,\mathfrak{m},k)\to(B,\mathfrak{n},l)$
For a (equivalently any) regular factorization of f, $A\to R\to\widehat{B}$, we have $\mathrm{H}_n(A,B,l)=0$ for all $n\geq 3$ and

$$0=\mathrm{d}(f):=\dim_l \mathrm{H}_2(R,\widehat{B},l)-\dim_l \mathrm{H}_1(R,\widehat{B},l)+\dim R-\dim B$$

Reasonably complete intersection (*rci*) $f\colon (A,\mathfrak{m},k)\to(B,\mathfrak{n},l)$
There exists a regular factorization of f, $A\to R\to\widehat{B}$, and a surjection of local rings $Q\to R$ such that

$$\mathrm{H}_n(Q,R,l)=0=\mathrm{H}_n(Q,\widehat{B},l)\quad\text{for all } n\geq 2.$$

Adequacy of the new classes

The classes of *rci* and *mci* homomorphisms **are ci-adequate**.

As for the class of *fci* homomorphisms, it satisfies conditions (A1) and (A2), and one implication of (A3), but we have not managed to give a proof for $f\colon A\to B$ *fci* and B CM implies A CM in full generality. However, we have obtained evidence pointing to its validity:

1. It holds for f essentially of finite type.
2. There exists a ci-adequate class of homomorphisms, that we will call *fci'*, which coincides with the *fci* class at least for essentially of finite type homomorphisms.

Adequacy of the new classes

The idea for the definition of the *fci'* is the following: for a surjective map $f\colon A\to B$ with $I=\ker(f)$, we say that f is *fci'* if $\mathrm{H}_1^{Kos}(I)$ is a free B-module and the complete intersection dimension $\mathrm{CI}\text{-}\dim_A(B)<\infty$.

In general, the definition of *fci'* extends the surjective case via regular factorizations: they are the homomorphisms for which there exists a regular factorization $A\to R\to\widehat{B}$ such that $\mathrm{H}_1^{Kos}(I)$ is a free $\widehat{B}$-module and $\mathrm{CI}\text{-}\dim_R(\widehat{B})<\infty$, where $I=\ker(R\to\widehat{B})$.

Comparison

$$\begin{tikzcd}
 & & mci \arrow[dr, Rightarrow] \arrow[dl, Rightarrow, shift left, "/" marking] & \\
ci \arrow[r, Rightarrow, shift left] & rci \arrow[l, Rightarrow, shift left, "/" marking] \arrow[ur, Rightarrow, shift left] \arrow[dr, Rightarrow] & & qci \arrow[dl, Rightarrow, shift left, "/" marking] \\
 & & fci \arrow[ur, Rightarrow, shift left] &
\end{tikzcd}$$

Comparison counterexamples

We will give some details for the counterexamples.

rci $\not\Rightarrow$ ***ci***. A local map between complete intersections is *rci* (but this fails for *ci*). Indeed, taking a Cohen factorization $A\to R\to\widehat{B}$, $R=\widehat{R}$ is a complete intersection by flat ascent, so R is a quotient of some regular ring Q by a regular sequence (so $\mathrm{H}_n(Q,R,l)=0$ for $n\geq 2$). As for the vanishing of $\mathrm{H}_n(Q,\widehat{B},l)$ for $n\geq 2$, this is a consequence of the fact that, for a local homomorphism $S\to T$ where S is regular and T complete intersection, $\mathrm{H}_n(S,T,-)=0$ for $n\geq 2$.
Both ***qci*** $\not\Rightarrow$ ***fci*** and ***mci*** $\not\Rightarrow$ ***rci*** follow from the same example. Avramov, Henriques and Şega showed that for the Artinian ring

$$R=\frac{k[w,x,y,z]}{(w^2,wx-y^2,wy-xz,wz,x^2+yz,xy,z^2)}\quad(\mathrm{char}(k)\neq 2)$$

there is an exact zero-divisor (x) and $\mathrm{CI}\text{-}\dim_R(R/(x))=\infty$. Such a quotient is always *qci*, but infinite complete intersection dimension means it is not *fci'* and thus not *fci*.
Because $\mathrm{H}_n(R,R/(x),-)=0$ for $n\geq 3$, $d(f)=\dim R-\dim R/(x)-\mathrm{grade}(xR)$, so $d(f)=0$ and it is *mci*. If it were *rci*, then it would also be *fci*, a contradiction.

Conclusions

- *rci* and *mci* homomorphisms constitue ci-adequate classes.
- The adequacy of *fci* homomorphisms is contingent on removing the essentially of finite type hypothesis for descent in (A3).
- Settling the relation between *mci* and *qci* would answer a conjecture of Avramov, Henriques and Şega.

Bibliography

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Sponsored by: PRIN 2022 (MUR grant 2022K48YYP)
Unirationality, Hilbert schemes and Singularities
Foundation Compositio Mathematica