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Unimodular rows over real algebraic varieties

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Introduction

Let R be a commutative Noetherian ring with unity.

- A row $(a_1, \ldots, a_n) \in R^n$ is called a **unimodular row** if there exists another row $(b_1, \ldots, b_n) \in R^n$ such that:
$$\sum_{i=1}^{n} a_i b_i = 1,$$
i.e., the ideal $\langle a_1, \ldots, a_n \rangle R = R$. The set of all unimodular rows of length n over R is denoted by $\mathrm{Um}_n(R)$.
- The group $\mathrm{GL}_n(R)$, and its subgroups $\mathrm{SL}_n(R)$ and $\mathrm{E}_n(R)$ acts on the set $\mathrm{Um}_n(R)$ of unimodular rows of length n. The orbit spaces of these group actions are denoted by $\frac{\mathrm{Um}_n(R)}{\mathrm{GL}_n(R)}$, $\frac{\mathrm{Um}_n(R)}{\mathrm{SL}_n(R)}$, and $\frac{\mathrm{Um}_n(R)}{\mathrm{E}_n(R)}$ respectively.

Main Question

Given a ring R and an integer n, is the orbit space $\frac{\mathrm{Um}_n(R)}{\mathrm{SL}_n(R)}$ trivial?

- When the ring R is a field, this question simply asks whether any non-zero vector in an n-dimensional R-vector space can be extended to a basis.
- For semi-local rings, the answer to this question is always affirmative.
- However, an affirmative answer is no longer guaranteed for arbitrary rings.

Literature survey

Let d be the Krull dimension of the ring R.

Bound	Author(s)	Structure of $\frac{\mathrm{Um}_n(R)}{\mathrm{SL}_n(R)}$
$n \geq d+2$	Bass	**Trivial** for any ring.
$n \leq d+1$	Swan	**Non-trivial** even for smooth affine algebras over $\mathbb{R}$.
$n = d+1$	Suslin	**Trivial** for affine algebras over perfect C_1-fields.
$n \leq d$	Mohan Kumar	**Non-trivial** for affine algebras over C_1-fields.
$n \leq d-1$	Mohan Kumar	**Non-trivial** for affine algebras over algebraically closed fields.
$n = d$	Fasel–Rao–Swan	**Trivial** for smooth affine algebras over algebraically closed fields.
$n = d+1$	Das–Tikader–Zinna	For a smooth real algebraic variety $X = \mathrm{Spec}(R)$ with $X(\mathbb{R}) \neq \varnothing$: $\frac{\mathrm{Um}_{d+1}(R)}{\mathrm{SL}_{d+1}(R)} \simeq \begin{cases} \bigoplus_{c \in \mathcal{C}} \mathbb{Z}, & d \text{ even} \\ \bigoplus_{c \leq \lvert\mathcal{C}\rvert} \mathbb{Z}/2\mathbb{Z}, & d \text{ odd} \end{cases}$ where $\mathcal{C}$ is the set of connected components of $X(\mathbb{R})$.

Main theorems

Theorem 1 [Banerjee 2025]

Let $X = \mathrm{Spec}(R)$ be an algebraic variety (not necessarily smooth) of dimension d over $\mathbb{R}$ such that either $X(\mathbb{R}) = \varnothing$ or the intersection of all real maximal ideals has height at least 1. Then the orbit space $\frac{\mathrm{Um}_{d+1}(R)}{\mathrm{SL}_{d+1}(R)}$ is trivial.

Theorem 2 [Banerjee–Fasel–Lerbet (in preparation)]

Let $X = \mathrm{Spec}(R)$ be a smooth algebraic variety of dimension d over $\mathbb{R}$ such that $X(\mathbb{R}) = \varnothing$. Then the orbit space $\frac{\mathrm{Um}_d(R)}{\mathrm{SL}_d(R)}$ is trivial.

Sketch of the proof of Theorem 2

- Let $v = (a_1, \ldots, a_d) \in \mathrm{Um_d(R)}$.
- A unimodular row of the form $(u_1^{n!}, u_2, \ldots, u_{n+1})$ is called a factorial row, and it is always completable to the first row of an invertible matrix by Suslin's factorial theorem.
- Therefore, it is enough to show that $v \equiv (c_1^{(d-1)!}, c_2, \ldots, c_d) \mod \mathrm{E_d(R)}$ for some $(c_1, c_2, \ldots, c_d) \in \mathrm{Um_d(R)}$.
- Using Swan's version of Bertini theorem one can find $\lambda_i, \mu_i, \kappa_i \in R$ such that $R/\langle b_4, \ldots, b_d \rangle$ is a smooth threefold with no real points, where $b_i = a_i + \lambda_i a_1 + \mu_i a_2 + \kappa_i a_3$.
- Let $v' = (a_1, a_2, a_3, b_4, \ldots, b_d)$. Then $v \equiv v' \mod \mathrm{E_d(R)}$.
- Therefore, we may replace v with v' and assume that $B := R/\langle a_4, \ldots, a_d \rangle$ is a smooth threefold with no real points.
- Let 'bar' denote going modulo $\langle a_4, \ldots, a_d \rangle$.
- Then there is a natural map $\frac{\mathrm{Um_3(B)}}{\mathrm{E_3(B)}} \to \frac{\mathrm{Um_d(R)}}{\mathrm{E_d(R)}}$ such that $(\overline{u}, \overline{v}, \overline{w}) \mapsto (u, v, w, a_4, \ldots, a_d)$.
- Then we prove that the Vaserstein symbol map $V : \frac{\mathrm{Um_3(B)}}{\mathrm{E_3(B)}} \to W_E(B)$ is an isomorphism, where $W_E(B)$ is the elementary symplectic Witt group of B.
- There is an isomorphism $W_E(B) \simeq GW^3_{1,\mathrm{red}}(B)$, where $GW^3_{1,\mathrm{red}}(B)$ is a reduced Grothendieck-Witt group of B.
- Then one can use the Gersten-Grothendieck-Witt spectral sequence $E(3)_1^{p,q}$ to show that $GW^3_{1,\mathrm{red}}(B) \simeq H^2(X, K_3^{MW})$, where $X = \mathrm{Spec}(B)$.
- Using the projection map and the fact that $X(\mathbb{R}) = \varnothing$ we prove that $H^2(X, K_3^{MW}) \simeq H^2(X, K_3^M)$.
- Then we compute $H^2(X, K_3^M)$ and show that it is a divisible group.
- Further we prove that $W_E(B)$ is a Mennicke symbol.
- The divisibility of $W_E(B)$ ensures the existence of another unimodular vector $(\overline{a}, \overline{b}, \overline{c}) \in \mathrm{Um_3(B)}$ such that $[(\overline{a}_1, \overline{a}_2, \overline{a}_3)] = [(\overline{a}, \overline{b}, \overline{c})]^{(d-1)!}$ in $W_E(B)$.
- Since $W_E(B)$ is a Mennicke symbol, we get that $[(\overline{a}, \overline{b}, \overline{c})]^{(d-1)!} = [(\overline{a}^{(d-1)!}, \overline{b}, \overline{c})]$ in $W_E(B)$.
- This will imply that $(a_1, a_2, a_3, a_4, \ldots, a_d) \equiv (a^{(d-1)!}, b, c, a_4, \ldots, a_d) \mod \mathrm{E_d(R)}$. □

Conclusions

- Theorem 1, together with the work of Das–Tikader–Zinna, completely classifies the structure of the orbit space $\frac{\mathrm{Um}_{d+1}(R)}{\mathrm{SL}_{d+1}(R)}$ for smooth d-dimensional real algebraic varieties.
- Theorem 2 establishes an analogue of Fasel–Rao–Swan over the base field $\mathbb{R}$.

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