Commutative Algebra and Algebraic Geometry in Milan

Politecnico di Milano, June 29–July 3, 2026

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Anti-Iitaka inequality in positive characteristic[BBC]

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Introduction

An invariant used to classify varieties is their Kodaira dimension $\kappa(X, K_X)$, measuring the "positivity" of the canonical divisor K_X.

Iitaka Conjecture $C_{n,m}$: Let $f : X \to Y$ be a fibration between smooth projective varieties over $\mathbb{C}$ and let X_y be a general fibre of f, then

$$\kappa(X, K_X) \geq \kappa(X_y, K_{X_y}) + \kappa(Y, K_Y).$$

Theorem $C^-_{n,m}$ [C,BBC]: Let $f\colon X \to Y$ be a fibration of smooth projective varieties over $\mathbb{C}$ and let X_y be a general fibre of f. Suppose that $\mathcal{J}(X_y; \|{-K_X}\|_y)$ is trivial. Then

$$\kappa(X, -K_X) \leq \kappa(X_y, -K_{X_y}) + \kappa(Y, -K_Y).$$

Main ingredients

Positivity descent: Assume $\kappa(X, -K_X) \geq 0$, then $\kappa(Y, -K_Y) \geq 0$.
Let $\Lambda \geq 0$ be a general $\mathbb{Q}$-divisor such that $\Lambda \sim_{\mathbb{Q}} -K_X$.
By Bertini theorems* (X_y, Λ_y) is klt.
By the Canonical Bundle Formula** we conclude that there exists a $\mathbb{Q}$-divisor B such that $K_X + \Lambda \sim_{\mathbb{Q}} 0 \sim_{\mathbb{Q}} f^*(K_Y + B)$, whence $-K_Y \sim_{\mathbb{Q}} B \geq 0$.

⚠ *Fails in char. $p > 0$ ⚠ **Not known in char. $p > 0$

Globally F-split varieties

A pair over an algebraically closed field of char. $p > 0$ (X, Δ), with $\Delta \geq 0$ a $\mathbb{Z}_{(p)}$-divisor is called **globally F-split** if there exists $e > 0$ such that the map $\mathcal{O}_X \to F^e_*\mathcal{O}_X((p^e - 1)\Delta)$ splits.

Examples: $\mathbb{P}^1$ is globally F-split, elliptic curves are globally F-split iff they are ORDINARY curves of higher genus are never globally F-split.

It is possible to define spaces that measure when a variety is globally F-split.
The existence of a splitting is equivalent to the surjectivity of the global sections of the trace map.

$$S^0(X) := \mathrm{Im}(T^e).$$

We define relative and perturbed versions:

$$S^0_y(X) := \mathrm{Im}(T^e_y),$$

and

$$Q^0_y(X) := \bigcap_{m\geq 0}\ \bigcap_{D\in|-mK_X|}\ \bigcap_{e_0>0}\ \sum_{e\geq e_0} \mathrm{Im}(T^e_{D/(p^e-1),y}).$$

Theorem $C^-_{n,m}$, char. $p > 0$

Let $f\colon X \to Y$ be a generically smooth fibration of smooth projective varieties, of dimensions n and m respectively, over an algebraically closed field k of characteristic $p > 0$, and let $y \in Y$ be a general point.
Assume

1. X_y is normal,
2. $Q^0_y(X) = k$,
3. there exists $\ell \geq 1$ not divisible by p such that $|-\ell K_X|$ induces the Iitaka fibration and $\mathrm{Bs}(-\ell K_X)$ does not dominate Y.

Then

$$\kappa(X, -K_X) \leq \kappa(X_y, -K_{X_y}) + \kappa(Y, -K_Y).$$

Furthermore, if $\kappa(Y, -K_Y) = 0$, equality holds.

Positivity descent in char. $p > 0$

Using Grothendieck duality applied to the Frobenius morphism, we have:

Theorem [SS]
If X is globally F-split, then there exists $\Lambda \geq 0$, $\Lambda \sim_{\mathbb{Q}} -K_X$ such that (X, Λ) is globally F-split.
Substitutes (*)

Theorem [DS]
If $f\colon X \to Y$ is a fibration such that

$$K_X + \Lambda \sim_{\mathbb{Z}_{(p)}} 0$$

and (X_y, Λ_y) is globally F-split, then there exists $B \geq 0$ such that

$$K_X + \Lambda \sim_{\mathbb{Z}_{(p)}} f^*(K_Y + B).$$

Substitutes (**)

Positivity descent

Let $f\colon X \to Y$ be a fibration between smooth projective varieties over an algebraically closed field k of char. $p > 0$ such that the general fibre X_y is normal and $S^0_y(X) = k$.
If $\kappa(X, -K_X) \geq 0$, then $\kappa(Y, -K_Y) \geq 0$.

If, additionally, we assume $Q^0_y(X) = k$, we obtain a "perturbed" version of positivity descent.

How restrictive are the assumptions?

Theorem [H,S] Let X be a projective variety over $\mathbb{C}$, then $\mathcal{J}(X)_p = \tau(X_p)$ for $p \gg 0$.
Theorem [SS] Let X be of log Fano type over $\mathbb{C}$. Then, for all but finitely many primes p, its reduction mod p is globally F-regular.
Weak Ordinarity Conjecture [SS] Let X be of log Calabi–Yau type over $\mathbb{C}$. Then there are infinitely many primes p such that its reduction mod p is globally F-split.

Our conjecture

Let $f\colon X \to Y$ be a fibration of smooth projective varieties over $\mathbb{C}$.
Assume $\mathcal{J}(X_y; \|{-K_X}\|_y)$ is trivial for $y \in Y$ general.
Denote by X_p the reduction mod p.
Then there are infinitely many primes p such that

$$Q^0_{y_p}(X_p) = H^0(X_p, \mathcal{O}_{X_p}).$$

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Sponsored by: MUR PRIN PRIN 2022 (MUR grant 2022K48YYP) **Unirationality, Hilbert schemes and Singularities** **Foundation Compositio Mathematica**