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Join-meet binomial algebras of distributive lattices

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Introduction

We consider the *join-meet binomials* f_{ij} of a lattice L on $x_1, \dots, x_n$:

$$f_{ij} = x_i x_j - (x_i \vee x_j)(x_i \wedge x_j), \ 1 \le i < j \le n.$$

The ideal $I_L = (f_{ij} : \quad 1 \le i < j \le n)$ has been widely studied since its introduction in [Hib87]. Using a different perspective, we focus on the *join-meet binomial algebra* of a simple distributive lattice L:

$$\mathcal{R}_K(L) = K[f_{ij} : \quad 1 \le i < j \le n].$$

Example. The coordinate rings of the Grassmannian $\mathrm{Gr}_K(2, m)$ is the join-meet algebra of the divisor lattice $D_{2 \cdot 3^{m-1}}$:

$$K[\mathrm{Gr}_K(2, m)] = \mathcal{R}_K(D_{2 \cdot 3^{m-1}}).$$

The join-meet binomials are precisely the 2-minors.

Algebras with straightening laws

Let $R = K[\gamma_1, \dots, \gamma_n]$ be a graded algebra and let P be a poset on $\gamma_1, \dots, \gamma_n$. A monomial $\gamma_1 \cdots \gamma_m \in R$ is *standard* if $\gamma_1 \le \cdots \le \gamma_m$ in P.

Def. R is an *algebra with straightening laws* [Eis80] on P over K if

- (ASL 1) The standard monomials are a K-basis of R.
- (ASL 2) If γ_i and γ_j are incomparable with *straightening relation*

$$\gamma_i \gamma_j = \sum_k r_k \gamma_{k_1} \gamma_{k_2} \cdots, \quad 0 \ne r_k \in K, \quad \gamma_{k_1} \le \gamma_{k_2} \le \cdots$$

then $\gamma_{k_1} \le \gamma_i, \gamma_j$ for every k.

Let L be a thin distributive lattice (as in Figure 1) and consider the polynomial ring $S = K[x_1, \dots, x_n, y_1, \dots, y_n]$. Fix a monomial order $<$ on S for which $\mathrm{in}_<(f_{ij}) = x_i y_j$ for all join-meet binomial f_{ij} of L. Consider the poset P_n on $[n]^2$ with $(i, j) \le (i', j')$ if $i \le i'$ and $j \le j'$ and let $Q_L \subseteq P_n$ be the sublattice consisting of the pairs (i, j) such that x_i and y_j are not comparable in L (Figure 2).

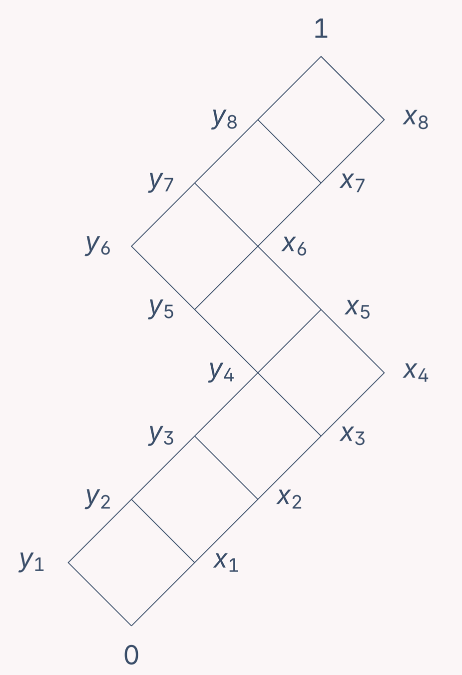

Figure 1. A thin distributive lattice L.

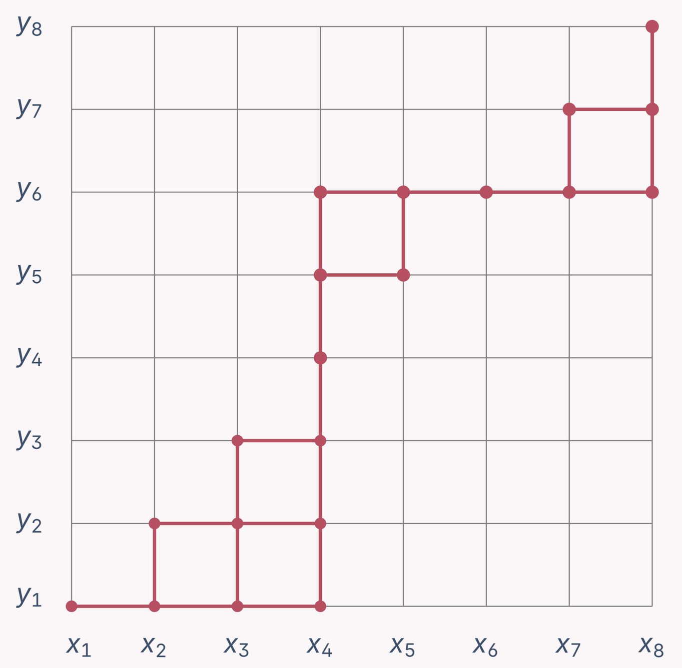

Figure 2. The sublattice Q_L of P_8.

Lemma [BH26] The subposet Q_L is a chain of P_n if and only if the divisor lattice $D_{2 \cdot 3^3}$ is not a sublattice of L.

Notice that there is an embedding $Q_L \hookrightarrow \mathcal{R}_K(L), \ (x_i, y_j) \mapsto f_{x_i, y_j}$.

Theorem [BH26]
The join-meet algebra $\mathcal{R}_K(L)$ of a thin distributive lattice L is a weakly ASL on Q_n over K, that is, (ASL 2) holds.

Proof idea.
We refer to Figure (1) to give the idea of the proof.
Consider the intervals $L_- = [0, x_5]$ and $L_+ = [x_3, 1]$ of L. Since L_- is the divisor lattice $D_{2 \cdot 3^4}$, the Plücker relations give the straightening laws on Q_{L_-}, which is the interval $[(x_1, y_1), (x_4, y_4)] \subseteq Q_L$. Hence $\mathcal{R}_K(L_-)$ is a weakly ASL on Q_{L_-}.
By induction on the rank, $\mathcal{R}_K(L_+)$ is a weakly ASL on Q_{L_+}, which is the interval $[(x_4, y_4), (x_8, y_8)] \subseteq Q_L$.
As (x_4, y_4) is an apex, no other straightening law appear for $\mathcal{R}_K(L)$.

Defining ideal of join-meet algebra

We study the defining ideal $I_{\mathcal{R}_K(L)}$ of $\mathcal{R}_K(L)$, that is the kernel of

$$\pi : K[x_{ij} : x_i \not\lessgtr x_j \text{ in } L] \twoheadrightarrow \mathcal{R}_K(L), \ x_{ij} \mapsto f_{ij}.$$

Example. The defining ideal of the join-meet algebra $\mathcal{R}_K(B_3)$ of the boolean lattice of rank 3 (Figure 3) is generated by two quadrics, that are independent of Plücker relations:

$$f_{3,6}f_{4,5} + f_{2,7}f_{4,6} - f_{3,5}f_{4,6} - f_{2,6}f_{4,7} = f_{2,7}f_{3,5} - f_{2,7}f_{4,6} + f_{2,6}f_{4,7} - f_{2,3}f_{5,7} = 0$$

Example. The divisor lattice $D_{2^2 \cdot 3^2}$ (Figure 4) has 9 binomial generators. These are the 2-minors of a 3×3 matrix, hence $I_{\mathcal{R}_K(D_{2^2 \cdot 3^2})} = 0$ and the join-meet algebra $\mathcal{R}_K(D_{2^2 \cdot 3^2})$ is a polynomial ring.

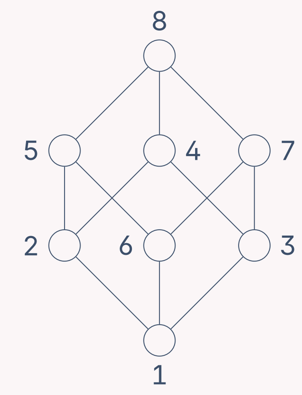

Figure 3. Boolean lattice B_3 of rank 3.

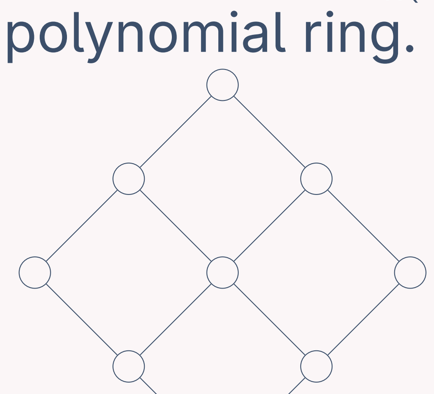

Figure 4. Divisor lattice $D_{2^2 \cdot 3^2}$.

Theorem [BH26]
The defining ideal of $\mathcal{R}_K(L)$ is zero if and only if L is planar and the divisor lattice $D_{2 \cdot 3^3}$ is not a sublattice of L.

Proof
If L is non-planar, then B_3 is a sublattice inducing relations among the join-meet binomials. The same happens if $D_{2 \cdot 3^3} \subseteq L$.
Assume L is planar and $D_{2 \cdot 3^3}$ is not a sublattice. If L is thin, then $\mathcal{R}_K(L)$ is a weakly ASL on a totally ordered set and there are no straightening relations. Otherwise, we have $L = D_{2^2 \cdot 3^2}$.

Quadratic relations

Theorem [BH26]
If L is a thin lattice, the defining ideal of $\mathcal{R}_K(L)$ has a quadratic Gröbner basis and hence it is Koszul. Moreover, $\mathcal{R}_K(L)$ is Gorenstein.

Theorem [BH26]
Let L be a planar lattice satisfying one of the following condition:

- (i) There are 4 elements with the same rank
- (ii) There are 3 elements with the same rank and other 3 elements covering them

Then the defining ideal $I_{\mathcal{R}_K(L)}$ of $\mathcal{R}_K(L)$ is not generated by quadrics.

Proof
(i) and (ii) guarantee that $D_{2^2 \cdot 3^3}$ is an interval of L. It follows that $\mathcal{R}_K(D_{2^2 \cdot 3^3}) \subseteq \mathcal{R}_K(L)$ is an algebra retract. By [OHH00], we have

$$\beta_{ij}(I_{\mathcal{R}_K(D_{2^2 \cdot 3^3})}) \le \beta_{ij}(I_{\mathcal{R}_K(L)}), \ \forall \, i, j.$$

By [BCV13], $I_{\mathcal{R}_K(D_{2^2 \cdot 3^3})}$ has minimal cubic relations and so does $I_{\mathcal{R}_K(L)}$.

Conjecture

Assume L to be planar. Then the defining ideal of $\mathcal{R}_K(L)$ is generated by quadrics if and only if at most one interval of L is $D_{2^2 \cdot 3^2}$.

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