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A level initial ideal of the 2-minors determinantal ideal

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Introduction

Let I be an ideal in a polynomial ring S with coefficient in a field $\Bbbk$. Fixing a term order $<$ on S, many properties are passed from $S/\mathrm{in}_<(I)$ to S/I, e.g. being Cohen-Macaulay, level or Gorenstein. It is quite natural to ask if some sort of converse holds.

From Conca and Varbaro ([3]), we know that, if $\mathrm{in}_<(I)$ is **squarefree**, then S/I is Cohen-Macaulay if and only if $S/\mathrm{in}_<(I)$ is Cohen-Macaulay. However, that does not hold for level (or Gorenstein).

Question: If S/I is **level** and admits an initial ideal $\mathrm{in}_{<'}(I)$ squarefree, then there exists another term order $<$ on S such that $\mathrm{in}_<(I)$ is squarefree and $S/\mathrm{in}_<(I)$ is level?

We answer the question positively when $I = I_2(X)$ is the **2-minors determinantal ideal** of a $m \times n$ matrix of variables $X = \{x_{i,j}\}$, with $i \in [m]$ and $j \in [n]$ $(m \le n)$, and $S = \Bbbk[X]$.

First of all, recall that a Noetherian standard graded $\Bbbk$-algebra R is said to be **level** if

(i) R is Cohen-Macaulay;

(ii) the socle of an (equivalently all) homogeneous Artinian reduction of R is concentrated in only one degree.

The right term order

The suitable term order $<$ for our goal is the degree reverse lexicographic order induced by the following order on the variables: to go from the highest to the lowest variable we scroll the rows of X from the left to the right but we leave in the last place the variables in the main diagonal of X.

For example, if $m = 3, n = 4$ we have

$$X = \begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} & x_{1,4} \\ x_{2,1} & x_{2,2} & x_{2,3} & x_{2,4} \\ x_{3,1} & x_{3,2} & x_{3,3} & x_{3,4} \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 & 12 & 11 & 10 \\ 9 & 2 & 8 & 7 \\ 6 & 5 & 1 & 4 \end{bmatrix}$$

In this case, the variables are ordered as it follows:

$$x_{3,3} < x_{2,2} < x_{1,1} < x_{3,4} < x_{3,2} < x_{3,1} <$$
$$< x_{2,4} < x_{2,3} < x_{2,1} < x_{1,4} < x_{1,3} < x_{1,2}.$$

From now on, $<$ will always be the term order we have just defined.

Remark

It is well-known that S/I is **level**. Moreover, from [1, Proposition 5.3.8], we know that the set of 2-minors is a Gröbner basis for I with respect to every degree reverse lexicographic term order, thus, since all the monomials that appear in a minor are squarefree, $\mathrm{in}_<(I)$ is **squarefree**.

Main theorem

With the notations above, $S/\mathrm{in}_<(I)$ is level.

Sketch of the proof (1)

Step 1. Since $\mathrm{in}_<(I)$ is squarefree, we can use the Stanley-Reisner correspondence and study the associated simplicial complex

$$\Delta := \{\sigma \in 2^{[m]\times[n]} : x_\sigma \notin \mathrm{in}_<(I)\}, \text{ where } x_\sigma := \prod_{(i,j)\in\sigma} x_{i,j}.$$

Sketch of the proof (2)

Step 2. We already know that Δ is Cohen-Macaulay. Moreover, we prove that every codimension-one face of Δ is contained, at most, in two facets. These two properties allow us to use two results of Gräbe ([5, Theorem 4] and [6, Hauptlemma 3.2]) which tell us about the structure of, $\omega_{S/\mathrm{in}_<(I)}$, the **canonical module** of $S/\mathrm{in}_<(I)$.

Step 3. Using the results of Gräbe and the combinatorial description of Δ, we are able to prove that all the minimal generators of $\omega_{S/\mathrm{in}_<(I)}$ (as a $\Bbbk$-vector space) live in degree $n - m$. One can prove that the latter property is actually equivalent to *(ii)* in the definition of level (see [4, Proposition 2.1.8(iv)]). Therefore are able to prove the levelness of $S/\mathrm{in}_<(I)$.

Shellability of Δ

We also prove that our simplicial complex Δ is shellable. The shellability is an important combinatorial property of a simplicial complex and it is often used to prove the Cohen-Macaulayness (in every characteristic).

A pure d-dimensional simplicial complex is said to be **shellable** if there exists a total order on its facets $F_1, \dots, F_c$ such that, for all $i \in \{2, \dots, c\}$, $\langle F_i \rangle \cap \langle F_1, \dots, F_{i-1} \rangle$ is a pure $(d-1)$-dimensional simplicial complex.

We generalize the construction in [2, Section 4], which covers the case $m = n$, to the case $m < n$.

Betti tables of initial ideals

Recall that if D is an homogeneous ideal in $T = \Bbbk[x_1, \dots, x_N]$, then for any initial ideal and any i, j we have

$$\beta_{i,j}(T/D) \le \beta_{i,j}(T/\mathrm{in}(D)).$$

If we consider our determinantal ideal I, we can ask if there exists some term order $<'$ such that $\beta_{i,j}(S/I) = \beta_{i,j}(S/\mathrm{in}_{<'}(I))$, for all i, j. Notice that Betti numbers can depend on $\mathrm{char}(\Bbbk)$, but in this case the fact that Betti tables coincide or not depends only on m and n and not on $\mathrm{char}(\Bbbk)$.
The only case not known was $m = 3$ and $n \ge 4$. We list every case.

- If $m = 2$, Betti tables coincide for any term order.
- If $m = n = 3$, Betti tables coincide if and only if $S/\mathrm{in}_{<'}(I)$ is level.
- If $m \ge 3$ and $n \ge 4$, Betti tables never coincide.

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