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Isotropic rank of harmonic polynomials

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1. Waring rank and apolarity theory

Given a form $F \in \mathcal{R} = \mathbb{C}[x_0 \dots x_n]_d$, its **Waring rank** is the number

$$\mathrm{rk}(F) = \min\{r \in \mathbb{N} \mid \exists \ell_1, \dots, \ell_r \in \mathcal{R}_1 \text{ s. t. } F = \ell_1^d + \dots + \ell_r^d\}.$$

Let us consider the ring $\mathcal{D} = \mathbb{C}[\alpha_0 \dots \alpha_n]$ and let

$$\circ : \mathcal{D} \times \mathcal{R} \to \mathcal{R}$$

be the apolar action of $\mathcal{D}$ on $\mathcal{R}$ by derivation. The ideal

$$\mathrm{Ann}(F) = \{G \in \mathcal{D} \mid G \circ F = 0\}$$

is called the **annihilator of** F. The **Apolarity Lemma** states that $F = \ell_1^d + \dots + \ell_r^d$ if and only, denoting by I the ideal of $[\ell_1], \dots, [\ell_r]$ as points of $\mathbb{P}^n$, we have that $I \subset \mathrm{Ann}(F)$.

2. Harmonic polynomials and isotropic rank

Let $\Delta = \alpha_0^2 + \dots + \alpha_n^2$ be the **Laplace operator**. The **space of harmonic polynomials** is the inverse system of Δ, i.e.

$$\mathcal{H} = (\Delta)^{-1} = \{F \in \mathcal{R} \mid \Delta \circ F = 0\}.$$

A linear form

$$\ell = a_0x_0 + \dots + a_nx_n \in \mathcal{R}_1$$

is called an **isotropic linear form** if $a_0^2 + \dots + a_n^2 = 0$. One can prove that

$$\mathcal{H}_d = \langle \ell^d \mid \ell \in \mathcal{R}_1 \text{ is isotropic} \rangle.$$

Hence, given $F \in \mathcal{H}_d$ we can define the **isotropic rank of** F as

$$\mathrm{irk}(F) = \min\{r \in \mathbb{N} \mid \exists \ell_1, \dots, \ell_r \in \mathcal{R}_1 \text{ isotropic s. t. } F = \ell_1^d + \dots + \ell_r^d\}.$$

If $F \in \mathcal{H}_d$ then, by definition, $\Delta \in \mathrm{Ann}(F)$ and finding an isotropic decomposition of F corresponds to finding an ideal of reduced points $I \subset \mathrm{Ann}(F)$ such that $\Delta \in I$. In general, it holds that

$$\mathrm{rk}(F) \leq \mathrm{irk}(F) \leq 2\,\mathrm{rk}(F)$$

and the bound is sharp. Determining the isotropic rank of a form F is a NP-hard problem, but we can manage it for some classes.

3. Generic isotropic rank

Consider the Veronese embedding

$$\begin{array}{rccc} \nu_d : & \mathbb{P}(\mathcal{R}_1) & \to & \mathbb{P}(\mathcal{R}_d) \\ & [\ell] & \mapsto & [\ell^d] \end{array}.$$

Since the isotropic linear forms are the points of $Q : q = 0 \subset \mathbb{P}(\mathcal{R}_1)$,

$$\mathrm{Isot}_d := \nu_d(Q) \subset \mathbb{P}(\mathcal{H}_d)$$

parameterises powers of isotropic linear forms and $\langle \mathrm{Isot}_d \rangle = \mathbb{P}(\mathcal{H}_d)$. The r-th secant variety of Isot_d is $\sigma_r(\mathrm{Isot}_d)$, where

$$\sigma_r^\circ(\mathrm{Isot}_d) = \bigcup_{[\ell_1], \dots, [\ell_r] \in \mathrm{Isot}_d} \langle [\ell_1], \dots, [\ell_r] \rangle, \quad \sigma_r = \overline{\sigma_r^\circ}.$$

If $r \in \mathbb{N}$ is the minimum such that $\sigma_r(\mathrm{Isot}_d) = \mathbb{P}(\mathcal{H}_d)$, then $\sigma_r^\circ(\mathrm{Isot}_d)$ is a Zariski dense open subset, i.e. the generic polynomial in $\mathcal{H}_d$ has isotropic rank r. Hence, to find the **generic isotropic rank** we have to compute

$$\min\{r \in \mathbb{N} \mid \dim \sigma_r(\mathrm{Isot}_d) = \dim \mathcal{H}_d - 1\}.$$

This computation can be translated in an interpolation problem as follows. Given a point $P \in Q$ defined by the ideal sheaf $\mathcal{I}_{P,Q}$, the **double point of** Q **supported at** P is the zero-dimensional scheme defined by the ideal sheaf $\mathcal{I}_{P,Q}^2$. It can be showed that

$$\mathrm{codim}\, \sigma_r(\mathrm{Isot}_d) = h^0(\mathcal{I}_{X,Q}(d))$$

where X is the union of r double points of Q in general position. Using the differential Horace method, we can determine $\dim \sigma_r(\mathrm{Isot}_d)$ for any $n, d, r \in \mathbb{N}$. In particular, we prove that:

- the isotropic rank of a generic harmonic polynomial of $\mathcal{H}_2$ is $n+1$;
- the isotropic rank of a generic harmonic polynomial of $\mathcal{H}_d$, with $d \geq 3$, is

$$\left\lceil \frac{1}{n}\left(\binom{n+d}{n} - \binom{n+d-2}{n}\right) \right\rceil.$$

4. Harmonic ternary forms

The ring of ternary harmonic forms $\mathcal{H} = \mathbb{C}[x_0, x_1, x_2]/(x_0^2 + x_1^2 + x_2^2)$ is isomorphic to the ring of binary forms of even degrees. Indeed, $Q : x_0^2 + x_1^2 + x_2^2 = 0$ is a conic, so

$$\mathrm{Isot}_d = \nu_d(Q) = (\nu_d(\nu_2(\mathbb{P}^1))) = \nu_{2d}(\mathbb{P}^1)$$

is a rational normal curve of degree $2d$. This reflects the canonical isomorphism of Lie algebras $\mathfrak{so}_3(\mathbb{C}) \cong \mathfrak{sl}_2(\mathbb{C})$. Using this isomorphism we can transform a harmonic ternary form in a binary form and then compute the rank by Sylvester algorithm. For instance:

$$\mathcal{H}_2 \ni (x_0 - ix_1)x_2 \to 2s^3t \in \mathcal{R}_4$$

has isotropic rank 4.

5. Harmonic monomials

A monomial of $\mathcal{R} = \mathbb{C}[x_0, \dots, x_n]$ is the product of powers of at most $n+1$ linearly independent forms. Given a harmonic monomial

$$\mathrm{m} = \ell_0^{a_0} \cdots \ell_p^{a_p} \in \mathcal{H}_d$$

with $\ell_i \neq \ell_j$, it is known that $\mathrm{rk}(\mathrm{m}) = (a_0+1)\cdots(a_p+1)$. Using $\boldsymbol{e}$-computability, we can prove that:

- $a_i \geq 2 \Rightarrow \ell_i$ is isotropic;
- if $a_0 = 1$, ℓ_0 is not isotropic and $\ell_1, \dots, \ell_p$ are isotropic, then $\mathrm{irk}(\mathrm{m}) = 2\,\mathrm{rk}(\mathrm{m})$, otherwise $\mathrm{irk}(\mathrm{m}) = \mathrm{rk}(\mathrm{m})$.

6. Harmonic quadrics

A quadric $F \in \mathcal{R}_2$ can be identified with a **symmetric** matrix A_F and asking $F \in \mathcal{H}_2$ is equivalent to ask $\mathrm{tr}(A_F) = 0$. For any $s \in \mathbb{N}$ and $\lambda \in \mathbb{C}$, let us consider the following matrices

$$J_s = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \in \mathbb{C}^{s,s}, \qquad K_s = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 \\ \vdots & & \iddots & \iddots & -1 \\ 0 & \iddots & \iddots & \iddots & 0 \\ 1 & \iddots & \iddots & & \vdots \\ 0 & -1 & 0 & \cdots & 0 \end{pmatrix} \in \mathbb{C}^{s,s}.$$

and $S_s(\lambda) = \lambda I_s + \frac{1}{2}J_s + \frac{i}{2}K_s$. Any symmetric complex matrix S is orthogonally similar to a unique diagonal block matrix

$$S_{s_1}(\lambda_1) \oplus \dots \oplus S_{s_r}(\lambda_r)$$

up to the order of the blocks. The sequence $(\lambda_1^{(s_1)}, \dots, \lambda_r^{(s_r)})$ is called the **normal sequence of** S. For a harmonic quadric $F \in \mathcal{H}_2$, we can prove that:

- A_F has normal form $(0^3, 0^2, \dots, 0^2) \Rightarrow \mathrm{irk}(F) = \mathrm{rk}(F) + 2$
- in all the other cases $\mathrm{rk}\,F = \mathrm{irk}\,F$.

7. Bibliography

1. S. Canino, C. Flavi, *Isotropic rank of harmonic polynomials*. Preprint available at https://arxiv.org/abs/2512.05195

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