Commutative Algebra and Algebraic Geometry in Milan

Politecnico di Milano, June 29–July 3, 2026

Laura Casabella   Alessio Sammartano

The variety of orthogonal frames

Goethe-Universität Frankfurt   casabell@math.uni-frankfurt.de

Introduction

An **orthogonal n-frame** is a n-tuple $(v_1, \dots, v_n) \in \Bbbk^d$ satisfying

$$\langle v_i, v_j \rangle = 0 \qquad (i \neq j).$$

- The set $\mathrm{V}(d,n) \subseteq \mathrm{Mat}(d,n)$ of all orthogonal n-frames is an algebraic variety, called the **variety of orthogonal frames**.

The ideal

$$I(d,n) = \left(\langle \underline{x}_j, \underline{x}_k \rangle : 1 \leq j < k \leq n\right) \subseteq \Bbbk[x_{i,j}]$$

defines $\mathrm{V}(d,n)$ set-theoretically.

We also define

$$R(d,n) = \Bbbk[x_{i,j}]/I(d,n), \quad X(d,n) = \mathrm{Spec} R(d,n).$$

Goal. Understanding $I(d,n)$ and $V(d,n)$.

$\mathrm{V}(d,n)$ and the Stiefel manifold

Adding the normalization equations $\langle v_i, v_i \rangle = 1$ yields the Stiefel manifold of *orthonormal* frames. Stiefel manifolds behave nicely (irreducible homogeneous spaces), while the varieties $V(d,n)$ are poorly understood (and typically reducible and singular).

Main theorem

$$D_{\mathrm{CI}}(n) := \min\left(2\left\lceil \tfrac{2n+1-\sqrt{8n+1}}{2}\right\rceil,\ 2\left\lceil \tfrac{2n-\sqrt{8n-7}-1}{2}\right\rceil + 1\right),$$

$$D_{\mathrm{prime}}(n) := \min\left(2\left\lfloor \tfrac{2n+1-\sqrt{8n+1}}{2}\right\rfloor + 2,\ 2\left\lfloor \tfrac{2n-\sqrt{8n-7}+1}{2}\right\rfloor + 1\right),$$

$$D_{\mathrm{UFD}}(n) := \min\left(2\left\lceil \tfrac{2n+1-\sqrt{8n-23}}{2}\right\rceil,\ 2\left\lceil \tfrac{2n-1-\sqrt{8n-31}}{2}\right\rceil + 1\right).$$

Theorem [1].

- $d \geq D_{\mathrm{CI}}(n) \Leftrightarrow I(d,n)$ is a complete intersection.
  In this case, $X(d,n)$ is reduced.
- $d \geq D_{\mathrm{prime}}(n) \Leftrightarrow I(d,n)$ is prime.
  In this case, $V(d,n)$ is normal.
- $d \geq D_{\mathrm{UFD}}(n) \Rightarrow R(d,n)$ is a UFD.

Stratification of $\mathrm{V}(d,n)$

- $p(A) = \#$ anisotropic columns of A,
- $q(A) = \#$ isotropic l.i. columns of A.
- $\mathcal{S}_{p,q} := \{A \in V(d,n) \mid p(A) = p,\ q(A) = q\}$.

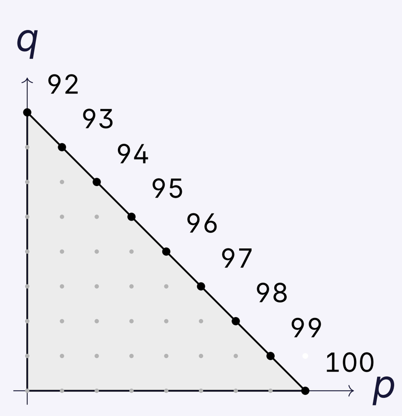
Then

$$V(d,n) = \bigsqcup_{p,q} \mathcal{S}_{p,q}.$$

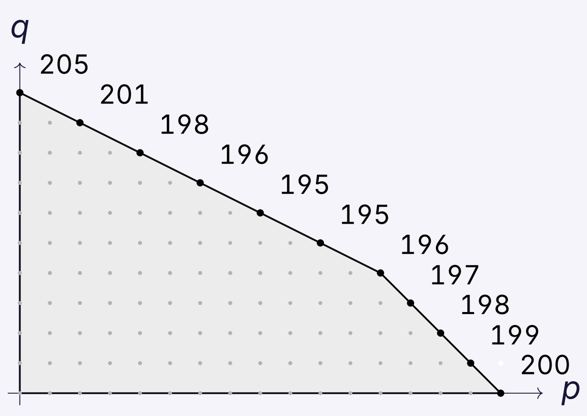
Theorem [1]. $\mathcal{S}_{p,q}$ is a smooth quasiprojective variety of dimension

$$\dim \mathcal{S}_{p,q} = pd + qd + qn - \tfrac{1}{2}p^2 - 2qp - \tfrac{3}{2}q^2 + \tfrac{1}{2}p - \tfrac{1}{2}q.$$

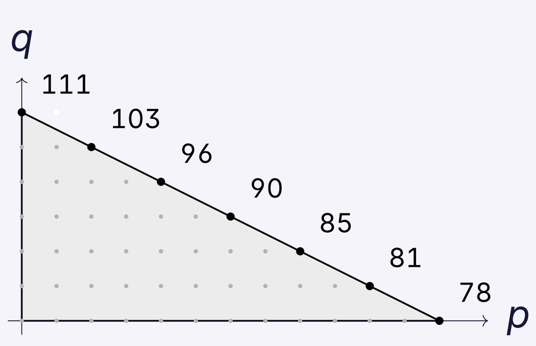
All connected components of $\mathcal{S}_{p,q}$ are irreducible and isomorphic.



$d = 16, n = 8$



$d = 20, n = 16$



$d = 12, n = 16$

Two distinguished components

Purely isotropic component. Assume $d \leq 2n$. Then

$$\mathrm{V}_{\mathrm{iso}}(d,n) = \overline{\mathcal{S}_{0,\lfloor d/2 \rfloor}}.$$

This variety is studied in [5].

- Defining ideal: $(q+1)$-minors $+ \{A^{\intercal}A = 0\}$.
- Components are normal, Cohen–Macaulay, with rational singularities.

Purely anisotropic component. Assume $d \geq n$. Define

$$\mathrm{V}_{\mathrm{ani}}(d,n) = \overline{\mathcal{S}_{n,0}}.$$

This variety is more obscure.

- Defining ideal: unknown.
- Smallest open case: $(d,n) = (6,6)$.

Example: $(d,n) = (6,4)$.

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ i & 0 & 0 & i \\ 0 & 1 & 0 & 3 \\ 0 & i & 0 & 3i \\ 0 & 0 & 1 & 5 \\ 0 & 0 & i & 5i \end{pmatrix} \in \mathcal{S}_{0,3} \qquad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \mathcal{S}_{4,0}$$

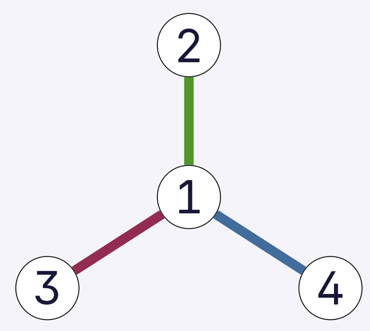
Lovász–Saks–Schrijver (LSS) ideals.

The **LSS ideal** of a graph $\mathcal{G}$ on n vertices is

$$\mathrm{LSS}(d,\mathcal{G}) = \left(\sum_{i=1}^{d} x_{i,j}x_{i,k} : (j,k) \text{ edge}\right),$$

cutting out *orthogonal representations* of $\mathcal{G}$ [4].

Example.



$$\begin{aligned} \mathrm{LSS}(d,\mathcal{G}) = (&x_{1,1}x_{2,1} + \cdots + x_{1,d}x_{2,d}, \\ &x_{1,1}x_{3,1} + \cdots + x_{1,d}x_{3,d}, \\ &x_{1,1}x_{4,1} + \cdots + x_{1,d}x_{4,d}) \end{aligned}$$

- For complete graphs, $\mathrm{LSS}(d, K_n) = I(d,n)$.

By [5], for fixed $d \gg n$, LSS ideals become well-behaved (complete intersection, prime, normal, UFD). We find the exact bounds for d.

Corollary [1].

- If $d \geq D_{\mathrm{CI}}(n)$, $\mathrm{LSS}(d,\mathcal{G})$ is a radical complete intersection.
- If $d \geq D_{\mathrm{prime}}(n)$, $\Bbbk[x_{i,j}]/\mathrm{LSS}(d,\mathcal{G})$ is a normal domain.
- If $d \geq D_{\mathrm{UFD}}(n)$, $\Bbbk[x_{i,j}]/\mathrm{LSS}(d,\mathcal{G})$ is a UFD.

These bounds are sharp in d.

Moreover, in [3] it is conjectured that $I(d,n)$ is radical.

Theorem [1]. $X(d,n)$ is generically reduced for all d, n.

Bibliography

[1] L. Casabella and A. Sammartano, *The variety of orthogonal frames*, arXiv:2512.25058.

[2] T. Ananyan and M. Hochster, *Small subalgebras of polynomial rings and Stillman's conjecture*, Journal of the American Mathematical Society 33 1 (2019), 291–309.

[3] A. Conca and V. Welker, *Lovász–Saks–Schrijver ideals and coordinate sections of determinantal varieties*, Algebra & Number Theory 13 (2) (2019), 455–484.

[4] L. Lovász, M. Saks and A. Schrijver, *Orthogonal representations and connectivity of graphs*, Linear Algebra and Applications 114/115 (1989), 439–454.

[5] J. Weyman, *Cohomology of Vector Bundles and Syzygies*, Cambridge University Press, 2003.

Sponsored by: MUR PRIN

PRIN 2022 (MUR grant 2022K48YYP)

Unirationality, Hilbert schemes and Singularities

Foundation Compositio Mathematica