Commutative Algebra and Algebraic Geometry in Milan

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Advances in asymptotic algebraic geometry

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Introduction

Segre–Veronese varieties and their secants are defined uniformly with respect to the base vector space V. As consequences, their description is independent of the dimension of V and enjoys the symmetry of the general linear group $\mathrm{GL}(V)$ on V.

For example, the second secant to the Segre variety on two factors $\sigma_2(S(V))$ (matrices of rank at most 2) is given by the elements in $V \otimes V$ of the form $v_1 \otimes w_1 + v_2 \otimes w_2$: we don't need to specify the dimension of the vector space V.

A GL-*variety* is an infinite-dimensional (i.e. non-Noetherian) GL-stable algebraic variety encoding a family $\mathcal{F}$ of varieties of tensors that are defined uniformly with respect to the base vector space. For a countable-dimensional vector space $\mathbf{V} = \bigcup_{i \in \mathbb{N}} k^i$, the image $(\mathbf{V}^* \times \mathbf{V}^*)^2 \to (\mathbf{V} \otimes \mathbf{V})^*$ sending (v_1, w_1, v_2, w_2) to $v_1 \otimes w_1 + v_2 \otimes w_2$ is the GL-variety associated with the family $\mathcal{F} = \{\sigma_2(S(k^i))\}_{i \in \mathbb{N}}$.

Recent results on GL-varieties show their tame behavior. A GL-variety X is:

1. defined by finitely many equations up to the action of GL (topological GL-Noetherianity), [5];
2. generically a finite product of GL-*spaces* (like $(\mathbf{V}^{\otimes d})^*$) up to a finite-dimensional factor, [1];
3. strongly unirational: a GL-space maps surjectively onto X, [2];
4. subject to Chevalley's theorem: image of a GL-constructible subset is GL-constructible, [1].

One of the most recent advances shows that a GL-variety encodes the singularities of a family of tensors: the singular locus of each member of the family can be read off from a finite amount of data, uniformly in the dimension. Moreover, the geometry of X intrinsically describes these data.

Main questions

Let X be a GL-variety sitting inside $(\mathbf{V}^{\otimes d})^*$ associated to a family of tensor varieties $\mathcal{F} = \{X_i\}_{i \in \mathbb{N}}$.

A) What is the relation between the family of $\mathcal{F}' = \{\mathrm{Sing}(X_i)\}_{i \in \mathbb{N}}$ and X? Are the singularities of $\mathcal{F}$ defined uniformly with respect to the base vector space? And does $\mathcal{F}'$ give rise to a GL-variety?

B) Can we define intrinsically a GL-subvariety Y of X whose associated family of tensor varieties coincides with $\mathcal{F}'$?

Examples

In the space $(\mathbf{V} \otimes \mathbf{V})^*$ of infinite-by-infinite matrices we look at the subspace X of rank-at-most-r elements for some fixed r. This GL-variety corresponds to the family $\{X_n\}_{n \in \mathbb{N}}$ of $n \times n$-matrices of rank at most r. For a fixed $n \geq r+1$, the singular locus $\mathrm{Sing}(X_n)$ of X_n is the space of $n \times n$-matrices of rank at most $r-1$. The GL-variety Y of infinite-by-infinite matrices of rank at most $r-1$ satisfies $Y_n = \mathrm{Sing}(X_n)$ for $n \geq r+1$, and it is the natural *singular locus* candidate. This framework generalizes to any GL-variety.

Singular points behave uniformly

Let X be a GL-variety. Let $\mathcal{F} = \{X_i\}$ be the family associated to X. Paper [3] shows that we can use the data of $\mathcal{F}' = \{\mathrm{Sing}(X_i)\}$ to construct a GL-variety $Y \subset X$, named the *singular locus* of X, such that $Y_i = \mathrm{Sing}(X_i)$ for i big enough. This means that we can describe the singularities of the family $\mathcal{F}$ with finitely many equations up to symmetry.

Main theorem

A local ring $(R, \mathfrak{m})$ is *regular* if the map:

$$\mathrm{Sym}(\mathfrak{m}/\mathfrak{m}^2) \to \bigoplus_{i \geq 0} \mathfrak{m}^i/\mathfrak{m}^{i+1}$$

is an isomorphism. A point $\mathfrak{p}$ sitting in a GL-variety $X = \mathrm{Spec}(A)$ (see below for details on A) is *regular* if $\mathcal{O}_{X,\mathfrak{p}}$ is a regular local ring, meaning that its tangent space and tangent cone coincide.

Paper [4] shows that the GL-variety Y defined in [3] coincides with the locus of points of X that are not regular, a notion merely depending on the geometry of X, without the action of GL and without passing to the finite-dimensional counterpart $\mathcal{F}$ of X.

Other results and future directions

Beyond smooth $\iff$ regular, [4] further proves smooth $\iff$ formally smooth $\iff$ jets lift $\iff$ module Ω_X is flat.

Some follow-up questions I am interested in:

1. Do finer (than regular/non-regular) invariants for singularity assemble into a GL-variety?
2. Is there a resolution of X?
3. Can the bound n ($Y_i = \mathrm{Sing}(X_i)\ \forall i > n$) be made effective?
4. Are there connections with ring-theoretic Noetherianity?

Affine GL-spaces and GL-varieties

The group GL is the infinite general linear group, where elements have the form:

$$\left(\begin{array}{c|c} g & 0 \\ \hline 0 & \mathbb{I} \end{array}\right),$$

where $g \in \mathrm{GL}(k^n)$ for some n, and $\mathbb{I}$ is the infinite identity matrix.

A *polynomial* GL-representation is a subquotient of a direct sum of tensor powers of $\mathbf{V}$.

Denote with S_λ the Schur functor associated to the partition λ (if $\lambda = (d)$ these are the symmetric d-way tensors in $\mathbf{V}^{\otimes d}$, if $\lambda = (1, \dots, 1)$ this representation is $\bigwedge^d(\mathbf{V})$). Then $S_\lambda(\mathbf{V})$ is a polynomial GL-representation.

A GL-*algebra* is an algebra over k equipped with an action of GL turning it into a polynomial representation. For example $\mathrm{Sym}(S_\lambda(\mathbf{V}))$ is a GL-algebra.

We focus on reduced *finite* GL-*generated* GL-algebras, namely, GL-algebras that can be generated by the GL-orbits of finitely many elements.

An affine GL-*variety* is the spectrum of a finite GL-generated GL-algebra. An affine GL-*space* is the spectrum of $\mathrm{Sym}(W)$ where W is a finite-length polynomial representation. Explicitly, the dual W^* of W coincides with the closed points of the corresponding GL-space.

Bibliography

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