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Auslander-Reiten annihilators

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Introduction

The Auslander-Reiten Conjecture for commutative Noetherian rings predicts that a finitely generated module is projective when certain Ext-modules vanish. But what if those Ext-modules do not vanish? We study the annihilators of these Ext-modules and formulate a generalisation of the Auslander-Reiten Conjecture.

The Auslander-Reiten Conjecture

- Let R be a commutative Noetherian local ring and M a finitely generated R-module. We learn in a first course in homological algebra that M is a projective (=free) module if and only if for any finitely generated module X and for any $i \geq 0$ one has $\mathrm{Ext}^i_R(M, X) = 0$.
- The Auslander-Reiten Conjecture predicts that instead of checking all finitely generated modules, it is enough to use $X = M \oplus R$.
- In other words, the Auslander-Reiten Conjecture states that if $\mathrm{Ext}^i_R(M, M \oplus R) = 0$ for all $i \geq 1$, then M has to be a free module.

Auslander-Reiten annihilators

Motivated by the Auslander-Reiten Conjecture, we consider the ideal

$$\mathrm{ARann}_R(M) := \bigcap_{i \geq 1} \mathrm{ann}_R \mathrm{Ext}^i_R(M, M \oplus R).$$

which we call the *Auslander-Reiten annihilator* of M. With this notation, the Auslander-Reiten Conjecture can be written as

$$M \text{ is free} \iff \mathrm{ARann}(M) = R \quad . \tag{1}$$

Stable annihilators

On the other hand, we consider another ideal

$$\underline{\mathrm{ann}}_R(M) := \bigcap_{i \geq 1} \bigcap_{X \in \mathrm{mod}(R)} \mathrm{ann}_R \mathrm{Ext}^i_R(M, X).$$

As mentioned above, it is well-known that

$$M \text{ is free} \iff \underline{\mathrm{ann}}_R(M) = R \quad . \tag{2}$$

We call $\underline{\mathrm{ann}}_R(M)$ the *stable annihilator* of M as it is equal to the annihilator of the stable endomorphism ring of M. We always have

$$\underline{\mathrm{ann}}_R(M) \subseteq \mathrm{ARann}_R(M)$$

Our question

- Combining (1) and (2), we can rewrite the Auslander-Reiten Conjecture as
$$\underline{\mathrm{ann}}_R(M) = R \iff \mathrm{ARann}(M) = R \quad .$$
- That is, for free modules, and only for free modules, the stable annihilator and the Auslander-Reiten annihilator agree and are both equal to R.
- What happens if we forget about the "and are both equal to R" part?
- So, our main question is: for an arbitrary finitely generated R-module M, do we have an equality
$$\underline{\mathrm{ann}}_R(M) = \mathrm{ARann}(M)?$$

Example: Hypersurfaces

Let $R = k[[x_1, \ldots, x_n]]/(f)$ be a hypersurface ring and M a maximal Cohen-Macaulay module. By the theory Tate cohomology of maximal Cohen-Macaulay modules over a Gorenstein ring and by the fact that one has 2-periodic resolutions over a hypersurface ring, we have an equality

$$\underline{\mathrm{ann}}_R(M) = \mathrm{ann}_R \mathrm{Ext}^2_R(M, M)$$

and this tells us that our question has an affirmative answer for maximal Cohen-Macaulay modules over a hypersurface rings because one, by definition, has an inclusion

$$\mathrm{ARann}_R(M) \subseteq \mathrm{ann}_R \mathrm{Ext}^2_R(M, M).$$

Theorem: Gorenstein isolated singularities

- We can prove the following theorem: Let R be a Gorenstein local ring which is an isolated singularity of Krull dimension at least 2. Then, our main question has an affirmative answer for every maximal Cohen-Macaulay module.
- Proof utilises Auslander-Reiten duality for isolated singularities and the following equality
$$\underline{\mathrm{ann}}_R(M) = \mathrm{ann}_R \mathrm{Ext}^{d+1}_R(\mathrm{Tr}(M), DM)$$
which works for all Cohen-Macaulay local rings with isolated singularities possessing a canonical module ω. Here, d is the Krull dimension, Tr is the Auslander transpose and $D = \mathrm{Hom}_R(-, \omega)$ is the canonical dual.

A counterexample

Our question does not always have a positive answer. Lyle and Maitra found a counterexample where R is a numerical semigroup ring and M is a canonical module.

Further on Auslander-Reiten annihilators

For more on Auslander-Reiten annihilators, check

https://www.sntp.ca/ozgur/arann

Bibliography

1. Maurice Auslander and Idun Reiten. On a generalized version of the Nakayama Conjecture. *Proc. Am. Math. Soc.* 52:69-74, 1975.
2. Maurice Auslander, Songqing Ding, and Øyvind Solberg. Liftings and weak liftings of modules. *J. Algebra*, 156(2):273–317, 1993.
3. Ragnar-Olaf Buchweitz. Maximal Cohen-Macaulay modules and Tate cohomology. *With appendices by Luchezar L. Avramov, Benjamin Briggs, Srikanth B. Iyengar and Janina C. Letz, volume 262 of Math. Surv. Monogr.* Providence, RI: American Mathematical Society (AMS), 2021.
4. David Eisenbud. Homological algebra on a complete intersection, with an application to group representations. *Trans. Amer. Math. Soc.*, 260(1):35–64, 1980.
5. Özgür Esentepe. Auslander–Reiten annihilators. *Collect. Math.* (2025). (published online)
6. Justin Lyle and Sarasij Maitra. Annihilators of (co)homology and their influence on the trace ideal. *preprint.* arXiv:2409.04686.

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