Commutative Algebra and Algebraic Geometry in Milan

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Weighted Veronese Rings via Convex Semigroups✿

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Veronese algebras

Commutative Algebra. Let $R = \mathbf{k}[x_1, \dots, x_n]$ be a non-standard graded polynomial ring with $\deg(x_i) = w_i \in \mathbb{Z}_{>0}$. For a fixed integer $d \geq 1$, the d-*th Veronese ring* of R is the subring defined as

$$V_{\mathbf{w},d} = \bigoplus_{j \geq 0} R_{dj}.$$

Algebraic Geometry. Consider the weighted projective space $\mathbb{P}(w_1) \times \cdots \times \mathbb{P}(w_n)$ over $\mathbf{k}$ and its structural sheaf $\mathcal{O}$. For every $d \geq 1$, the complete linear system

$$\mathbb{P}(w_1) \times \cdots \times \mathbb{P}(w_n) \xrightarrow{|\mathcal{O}(d)|} \mathbb{P}^N$$

maps into a weighted projective space. In particular, the coordinate ring of the image is $R/V_{\mathbf{w},d}$.

Invariant Theory. Suppose $\mathbf{k}$ contains a primitive d-th root ξ of unity, and consider the natural action of the cyclic group $\Gamma = \langle \xi \rangle$. It acts on R by scalar multiplication $x_i \mapsto \xi^{w_i} x_i$. Then the ring of invariants R^Γ is exactly $V_{\mathbf{w},d}$.

Affine semigroup rings

A *convex semigroup* is a subsemigroup of $\mathbb{N}^2$ generated by a convex sequence $\mathbf{c} \subset \mathbb{N}^2$, i.e. $2\mathbf{c}_i \leq \mathbf{c}_{i-1} + \mathbf{c}_{i+1}$ component-wise. We say that $\mathbf{c}_i$ is a *corner point* if $2\mathbf{c}_i < \mathbf{c}_{i-1} + \mathbf{c}_{i+1}$. A *convex semigroup ring* is the $\mathbf{k}$-subalgebra of $\mathbf{k}[x, y]$ generated by the monomials corresponding to $\mathbf{c}$, namely

$$S_{\mathbf{c}} = \mathbf{k}\left[x^{a_i} y^{b_i} \mid \mathbf{c} = \{(a_i, b_i)\}_{i=1}^{s}\right] \subset \mathbf{k}[x, y]. \quad (1)$$

An affine semigroup ring is a monomial subring of $R = \mathbf{k}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. We say that an affine semigroup $S \subseteq R$ is *closed under monomial division* if given $\mu_1, \mu_2 \in S$ with $\mu_1/\mu_2 \in R$, then $\mu_1/\mu_2 \in S$. Every affine semigroup closed under monomial division is **normal**.

Main theorem

Let $\mathbf{c} = \{c_i = (a_i, b_i)\}_{i=1}^{s} \subset \mathbb{N}^2$ be a convex sequence and let $S_{\mathbf{c}}$ be the associated convex semigroup ring. Assume that $S_{\mathbf{c}}$ is closed under monomial division in $\mathbf{k}[x, y]$. Then

(1) $S_{\mathbf{c}}$ has a determinantal presentation $S_{\mathbf{c}} = \mathbf{k}[t_1, \dots, t_s]/J$, where J is the ideal of 2×2 minors of a matrix $A_{\mathbf{c}}(\mathbf{t})$ whose entries are nonconstant monomials in the variables $t_1, \dots, t_s$;

(2) the total and graded Betti numbers of $S_{\mathbf{c}}$ over $\mathbf{k}[t_1, \dots, t_s]$ are given by

$$\beta_i^R(S_{\mathbf{c}}) = i \binom{s-1}{i+1} \text{ (they depend only on } s\text{); } \textit{and},$$

$$\beta_{i,j}^R(S_{\mathbf{c}}) = \sum_{\substack{1 \leq g_1 < \cdots < g_i \leq s, \\ \sum_{\ell=1}^{i} a_{g_\ell} + b_{g_\ell} = j}} \left| \{ \ell \mid g_{\ell+1} - g_\ell > 1 \} \right|.$$

Moreover, the Hilbert series of $S_{\mathbf{c}}$ depends only on the sums $a_i + b_i$ of the minimal algebra generators;

(3) the minimal generators of J are the 2×2 minors of $A_{\mathbf{c}}(\mathbf{t})$ given by submatrices whose main diagonal entries are variables. They form a Gröbner basis under the lexicographic order induced by $t_1 > t_2 > \cdots > t_n$;

(4) $S_{\mathbf{c}}$ is a normal, Cohen–Macaulay, Koszul algebra;

(5) the Betti numbers of the residue field $\mathbf{k}$ over $S_{\mathbf{c}}$ are given by

$$\beta_i^{S_{\mathbf{c}}}(\mathbf{k}) = (s-2)^{i-2}(s-1)^2 \text{ for } i \geq 2. \text{ [2]}$$

Sketch of the proof

One needs to prove that $\mathrm{gr}_{\mathfrak{m}}(S_{\mathbf{c}}) \cong \mathcal{F}(I)$, where

$$I = (x^{a_i} y^{b_i} \mid 1 \leq i \leq s), \quad \mathfrak{m} = I \cap S_{\mathbf{c}}, \quad \mathcal{F}(I) := \bigoplus_{j \geq 0} \frac{I^j}{\mathfrak{m}_{\mathbf{k}[x,y]} I^j}.$$

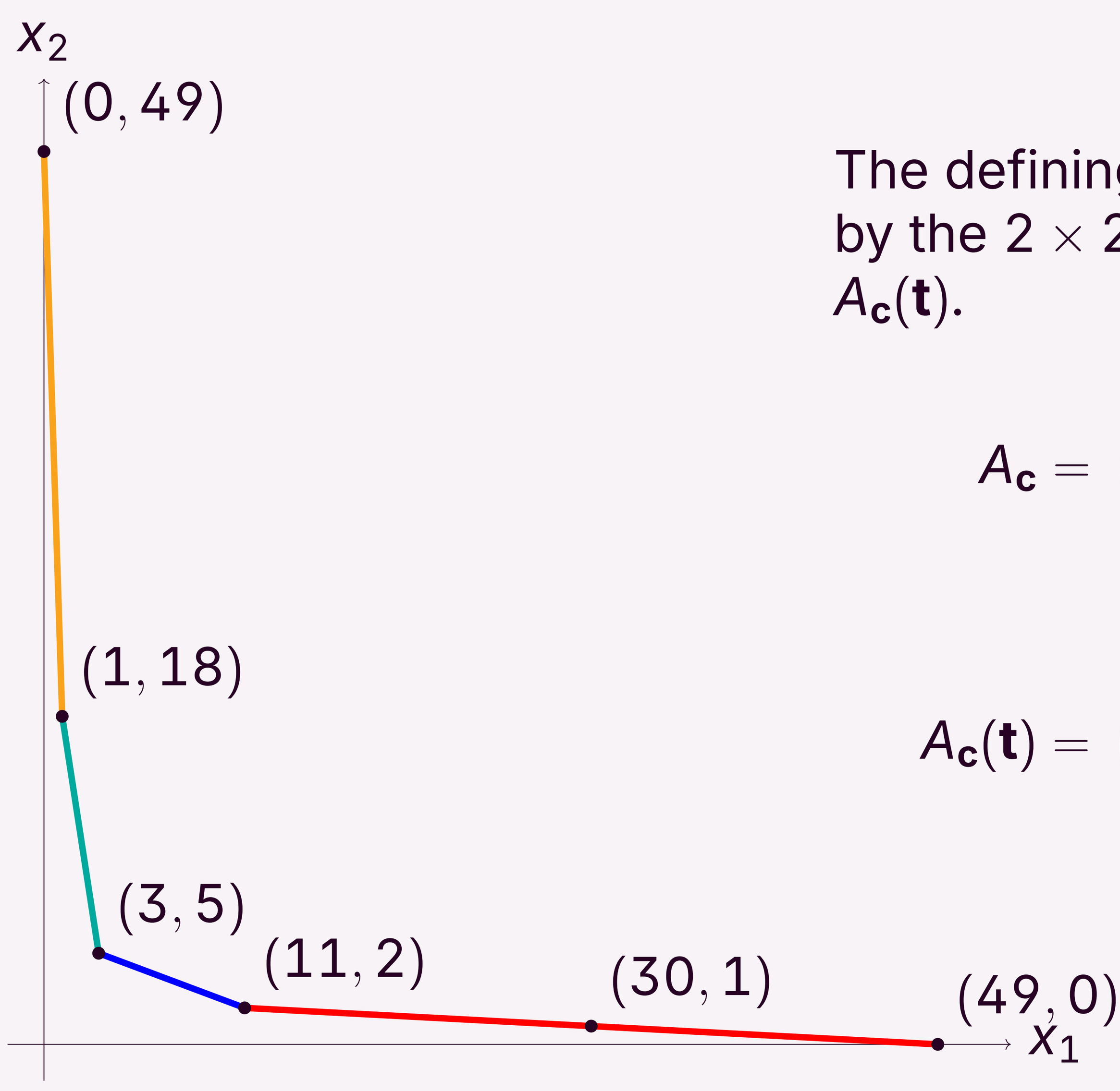
Following [1], one lifts information and formulae for Betti numbers, Hilbert function, and the Cohen-Macaulay property from $\mathcal{F}(I)$ to $S_{\mathbf{c}}$. The matrix $A_{\mathbf{c}}(\mathbf{t})$ can be explicit constructed using the corner points, and one sees explicitly the double inclusion between the 2-minors and the defining ideal of $S_{\mathbf{c}}$.

Linking result

Every 2-dimensional Veronese ring is a convex semigroup closed under monomial division. Conversely, every normal affine semigroup is isomorphic, but usually not graded isomorphic, to a 2-dimensional Veronese.

Matrix construction

We consider $V_{(31,1),49} = \mathbf{k}[x_2^{49}, x_1 x_2^{18}, x_1^3 x_2^5, x_1^{11} x_2^2, x_1^{30} x_2, x_1^{49}]$.



The defining ideal is generated by the 2×2 minors of the matrix $A_{\mathbf{c}}(\mathbf{t})$.

$$A_{\mathbf{c}} = \begin{pmatrix} t_1 & t_2 & & \\ & t_3 & t_4 & t_5 \\ & & t_5 & t_6 \end{pmatrix}$$

$$A_{\mathbf{c}}(\mathbf{t}) = \begin{pmatrix} t_1 & t_2 & t_3^3 & t_3^2 t_4^2 \\ t_2^2 & t_3 & t_4 & t_5 \\ t_2 t_3^3 t_4 & t_4^2 & t_5 & t_6 \end{pmatrix}$$

Non-examples in higher dimension

✦The Veronese $V_{(1,6,15),10}$ is not a convex semigroup.
✦The ring $V_{(3,4,5),6}$ does not admit a determinantal presentation.
✦The Veronese $V_{(3,4,5),15}$ is not Koszul. More in general, the following families of Veronese rings are not Koszul:

1. $V_{(3,3k+1,3k+2,w_4,\dots,w_n),3(3k+2)}$ for any integers $w_4, \dots, w_n > 3(3k+2)$ and $k \geq 1$;
2. $V_{(f+1,f+2,f^2+f-1,w_4,\dots,w_n),3(f^2+f+1)}$ for any integers $w_4, \dots, w_n > 3(f^2 + f + 1)$ and $f \geq 2$.

Bibliography

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