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Chain algebras of finite distributive lattices

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Preliminaries

- Let P be a finite poset. An *ideal* of P is a downward closed subset I of P.
- The set of all ideals of P, partially ordered by inclusion, is a finite distributive lattice (FDL). We will denote it by $L = \mathcal{J}(P)$.
- According to a classical result of Birkhoff, any FDL arises this way.
- The *width* of a poset is the number of elements in the largest antichain of P. By Dilworth's theorem, it is also the smallest number of disjoint chains needed to cover P.
- Let $L = \mathcal{J}(P)$. Then we define the *dimension* of L to be the width of P. We will say that L is *planar* if $\dim(L) \le 2$.
- Let $L = \mathcal{J}(P)$. The *Hibi ring* of L is the toric ring $K[\mathcal{H}_L] := K[T_a \mid a \in L]/(T_aT_b - T_{a\wedge b}T_{a\vee b} \mid a, b \in L)$. For a distributive lattice L many algebraic invariants of its Hibi ring can be computed in terms of combinatorics of L and P: Hilbert function/series, regularity, dimension, a-invariant, multiplicity.

What is a chain algebra?

Let L be an FDL with a labeling $t_0, t_1, \ldots, t_{|L|-1}$ on its elements. To each maximal chain C we associate a monomial t_C, which is the product of the corresponding labels.

Definition

The *chain algebra* is the toric algebra generated by all such monomials, in other words,
$K[\mathcal{C}_L] := K[\{t_C \mid C \text{ is a maximal chain of } L\}]$.
The goal is to study algebraic properties of $K[\mathcal{C}_L]$ and its defining ideal, and connect them to combinatorial properties of L and P.

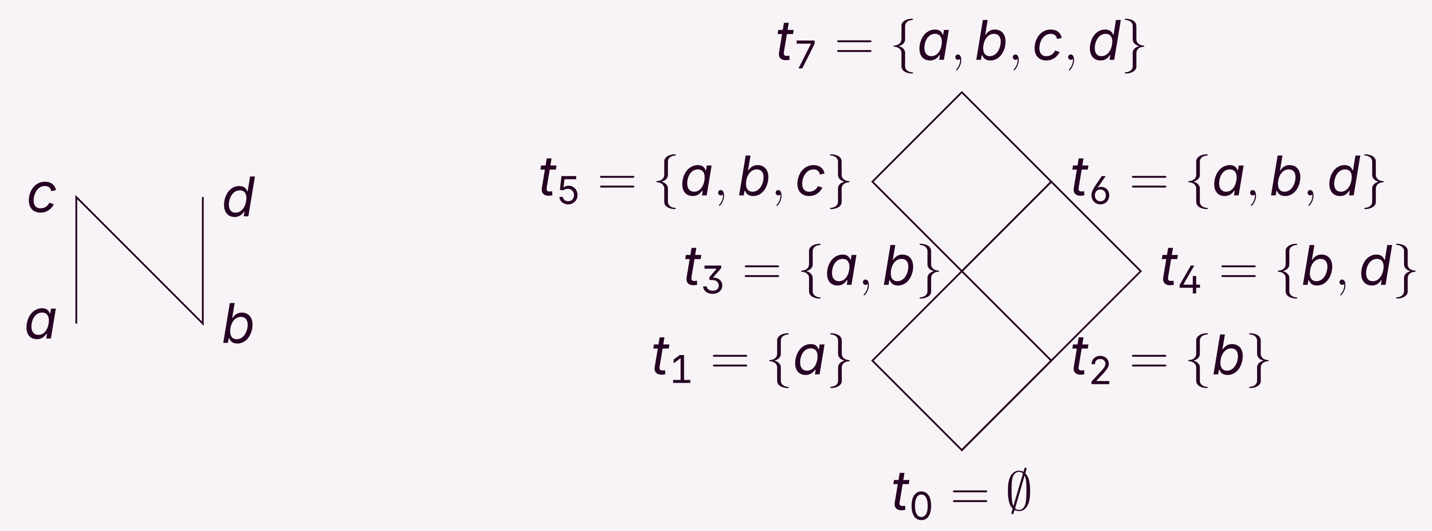


Figure: P and $L = \mathcal{J}(P)$

Example

In this example $w(P) = \dim(L) = 2$, therefore, L is a planar lattice.
The chain algebra equals $K[\mathcal{C}_L] = K[t_1t_3t_5, t_1t_3t_6, t_2t_3t_5, t_2t_3t_6, t_2t_4t_6]$.
One can check that $K[\mathcal{C}_L] \cong K[T_1, T_2, T_3, T_4, T_5]/(T_2T_3 - T_1T_4)$.

Sortable sets of monomials

Consider the following set of monomials (from the previous $K[\mathcal{C}_L]$):
$S = \{m_1 = t_1t_3t_5, m_2 = t_1t_3t_6, m_3 = t_2t_3t_5, m_4 = t_2t_3t_6, m_5 = t_2t_4t_6\}$.
$m_2m_3 = \underline{t_1}\cdot\underline{t_2}\cdot\underline{t_3}\cdot\underline{t_3}\cdot\underline{t_5}\cdot\underline{t_6} \Rightarrow (m_1, m_4) = sort(m_2, m_3) \leftarrow$ unsorted pair;
$m_2m_1 = \underline{t_1}\cdot\underline{t_1}\cdot\underline{t_3}\cdot\underline{t_3}\cdot\underline{t_5}\cdot\underline{t_6} \Rightarrow (m_1, m_2) = sort(m_2, m_1) \leftarrow$ "sorted" pair;
We have $(m_1, m_2) = sort(m_2, m_1) = sort(m_1, m_2) \leftarrow$ sorted pair.

Definition

A set of monomials S is called *sortable* if $sort(S \times S) \subseteq S \times S$.

Theorem (Sturmfels)

If S is a sortable set, then the defining ideal of $K[S]$ is generated by sorting relations, which, moreover, form a Gröbner basis for this ideal.

Sortable lattices and the main result

Let $L = \mathcal{J}(P)$, as before. We will say that L is *sortable* if there exists a labeling on L such that $S = \{t_C \mid C \text{ is a maximal chain of } L\}$ is a sortable set of monomials.

Theorem (Main result)

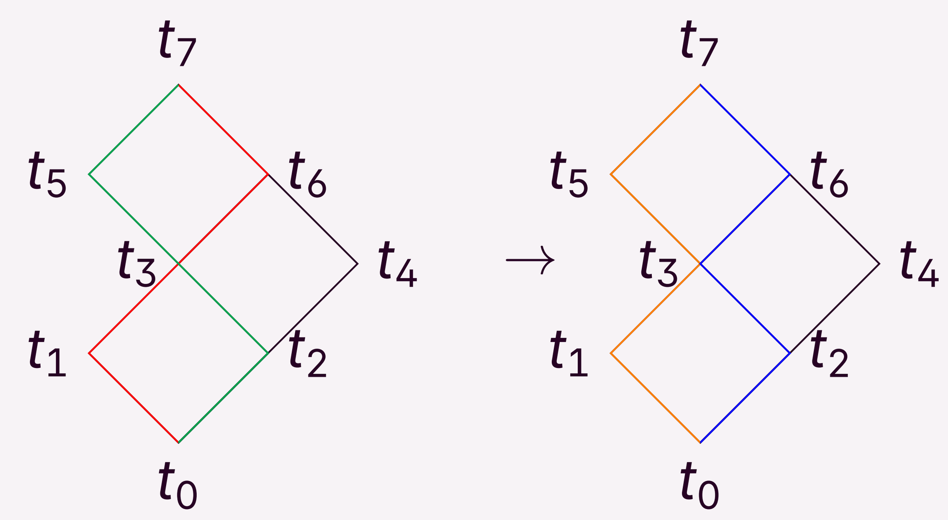
Let $L = \mathcal{J}(P)$ as before. The following are equivalent:

1. *L is planar;*
2. *L is sortable;*
3. *the defining ideal of $K[\mathcal{C}_L]$ has a quadratic Gröbner basis;*
4. *the defining ideal of $K[\mathcal{C}_L]$ is quadratically generated;*
5. *$K[\mathcal{C}_L]$ is a Hibi ring.*

Strategy of the proof: $1) \Rightarrow 2) \xRightarrow{\text{Sturmfels}} 3) \xRightarrow{\text{obvious}} 4) \Rightarrow 1)$;
$1) \Rightarrow 5) \xRightarrow{\text{by definition}} 4)$.

Proof of 1) ⇒ 2): resolving crossings

Consider the bottom-to-top, left-to-right labeling. One can show that a pair of monomials is unsorted iff the corresponding maximal chains traverse, and the sorting relations resolve these crossings:



Here, $sort(t_{C_2}, t_{C_3}) = (t_{C_1}, t_{C_4})$.
We will say that $C_1 = \min(C_2, C_3)$, $C_4 = \max(C_2, C_3)$.

Proof of 1) ⇒ 5)

We will introduce a partial order on maximal chains of a planar L. If two different chains C and D are non-traversing, then one of them is weakly to the left of the other one, say, C is to the left of D. Then we say $C \prec_{ch} D$. If C and D traverse each other, we declare them incomparable.

Theorem

With respect to the above partial order, maximal chains of a planar L form an FDL (call it L') with $C \wedge D = \min(C, D)$ and $C \vee D = \max(C, D)$. In fact, L' is an interval in the Young's lattice. Moreover, $K[\mathcal{C}_L] \cong K[\mathcal{H}_{L'}]$, and the sorting relations are exactly the Hibi relations.

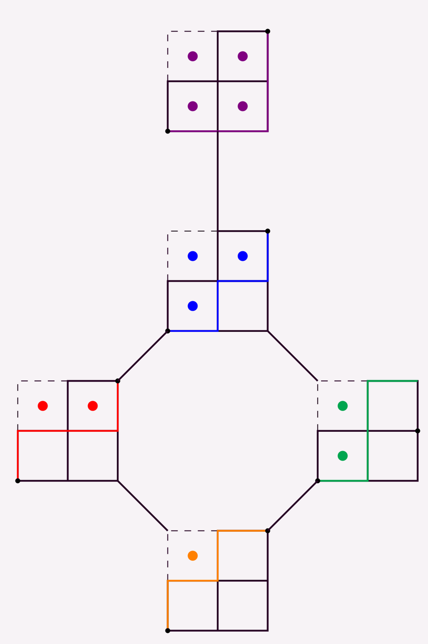


Figure: The lattice L' of maximal chains of L

Example

Let L be the same planar FDL as before. We consider its embedding into a rectangle, which is always possible for a planar FDL. Then each chain can be identified with a Young diagram it cuts from the upper left corner of this rectangle. We have $K[\mathcal{H}_{L'}] = K[T_1, T_2, T_3, T_4, T_5]/(T_2T_3 - T_1T_4) \cong K[\mathcal{C}_L]$.

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