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Bounds for Regularity of Toric Varieties in Terms of Multiplicity

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The Eisenbud-Goto Conjecture

Let S be a polynomial ring over an algebraically closed field. All primes P will be assumed to be homogeneous and contain no linear forms. Let $\operatorname{reg}(S/P)$ be the Castelnuovo-Mumford regularity of S/P, and $e(S/P)$ be the degree of the variety of P. Assume $e(S/P) \geq 2$.

Conjecture (Eisenbud-Goto, '84). P satisfies

$$\operatorname{reg}(S/P) \leq e(S/P) - \operatorname{height}(S/P) + 1.$$

Theorem ([1]). There are counterexamples to the Eisenbud-Goto conjecture; in fact, there is no polynomial function of the multiplicity which bounds the regularity of an arbitrary prime.

An Extremely Large Bound

The best known bound for regularity in terms of multiplicity for arbitrary primes is as follows.

Theorem ([2]). A prime P satisfies,

$$\operatorname{reg}(S/P) \leq (e^2) \uparrow^{e^4-1} (e+2).$$

Here, $\uparrow^{e^4-1}$ denotes e^4-1 copies of Knuth's up-arrow, which cannot be computed using for-loops and standard arithmetic operations.

Toric Ideals

For $\mathbf{b} \in \mathbb{Z}^n$, define $f_{\mathbf{b}} = x^{\mathbf{b}^+} - x^{\mathbf{b}^-}$, where $\mathbf{b}^+$ has ith coordinate $\max\{\mathbf{b}_i, 0\}$ and $\mathbf{b}^- = \mathbf{b}^+ - \mathbf{b}$. For $A \in \mathbb{Z}^{d\times n}$, the *toric ideal* of A is

$$P_A = \{f_{\mathbf{b}} : \mathbf{b} \in \ker A \cap \mathbb{Z}^n\}.$$

We want to know whether the Eisenbud-Goto conjecture holds for toric ideals; there is, however, no bound on the regularity of toric ideals in terms of their multiplicity that is better than the one above.

A General Bound for Toric Ideals

Main Theorem. Suppose that P is generated by polynomials with at most c terms (the case $c = 2$ includes toric ideals). For $\beta = c^{\frac{e(e^{h+1}-1)}{e-1}}$, P satisfies

$$\operatorname{reg}(S/P) \leq (e-1)^{2^{\beta-2}+1}.$$

If P is toric, then

$$\operatorname{reg}(S/P) \leq \frac{e^3(e^e-1)}{e-1}.$$

Sketch of the proof

Let $h = \operatorname{height}(P)$. We construct a homogeneous regular sequence $f_1, \ldots, f_h$ that occurs in at most β variables with $\deg(f_i) = e$ for all i. The key step uses induction on h and the prime avoidance lemma for vector spaces to find an element of degree e extending the regular sequence. There exists f of degree $(e-1)(h-1)$ such that $P = (f_1, \ldots, f_h) : f$ (see [3]), and so a short exact sequence

$$0 \to S/P(-(e-1)(h-1)) \to S/(f_1, \ldots, f_h) \to S/(f_1, \ldots, f_h, f) \to 0.$$

Thus, $\operatorname{reg}(S/P)$ is determined by $\operatorname{reg}(S/(f_1, \ldots, f_h))$, which is at most

$$(e-1)(h-1)^{2^{\beta-2}},$$

(see [4]) and the result follows from $h \leq e$ ([2]). If P is toric, then P satisfies

$$\operatorname{reg}(S/P) \leq n \cdot e \cdot h$$

(see [5]), and the result follows from $n \leq b$ and $h \leq e$.

Convex Geometry and a Second Bound

Theorem. For any toric ideal P_A, $\operatorname{reg}(S/P_A) \leq 2^e e^{5e}$.

Proof Sketch

We again use that the regularity is bounded above by $n \cdot e \cdot h$. If $\mathbf{b}_1, \ldots, \mathbf{b}_h$ forms a $\mathbb{Z}$-basis of $\ker(A) \cap \mathbb{Z}^n$, then

$$(f_{\mathbf{b}_1}, \ldots, f_{\mathbf{b}_h}) : \left(\prod_{i\leq n} x_i\right)^\infty = P_A.$$

As such, the number of variables in P_A equals the number of variables appearing in the $f_{\mathbf{b}_i}$. The number of variables appearing in $f_{\mathbf{b}_i}$ is at most $2\deg(f_{\mathbf{b}_i}) = ||\mathbf{b}_i||_1$ (using P_A homogeneous). Results from lattice theory yield a $\mathbb{Z}$-basis for $\ker(A)$ with small 1-norms.

Minkowski's Second Theorem

Theorem. For each $i \leq \dim(\ker(A)) =: h$, let

$$\lambda_i = \min\{r : \operatorname{span}(B_{0,r} \cap \ker(A)) \geq i\}.$$

Then

$$\left(\prod_{i\leq h} \lambda_i\right) \operatorname{vol}(B_{0,1}) \leq 2^h \operatorname{vol}(\mathbb{R}^h / \ker(A)).$$

Existence of a Small Basis

Proposition. $\ker(A)$ has a $\mathbb{Z}$=basis $\mathbf{b}_1, \ldots, \mathbf{b}_h$ with $||\mathbf{b}_i||_1 \leq h\lambda_h$.

Kushnirenko's Theorem

Theorem. The normalized volume of $\operatorname{conv}(a_1, \ldots, a_n)$ inside the lattice $\mathbb{Z}(a_2 - a_1, \ldots, a_n - a_1)$ equals $e(S/P_A)$.

Proof Sketch (continued)

Because $\ker(A)$ is an integer lattice, each $\lambda_i \in \mathbb{Z}$, so

$$\lambda_h \leq \left(\prod_{i\leq h} \lambda_i\right) \leq 2^h \operatorname{vol}(\mathbb{R}^h / \ker(A)) / \operatorname{vol}(B_{0,1}).$$

Smith Normal Form calculations yield that this value is at most 2^h times the normalized volume of $\operatorname{conv}(a_1, \ldots, a_n)$. Thus, each $f_{\mathbf{b}_i}$ can use at most $2^h he$ variables, and so P_A uses at most $2^h h^2 e$ variables.

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