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Jakub Jagiełła joint with Joachim Jelisiejew

Resolution of singularities of small secant varieties of Segre varieties

University of Warsaw j.jagiella@uw.edu.pl

1. Introduction

Let $V_1, \dots, V_d$ be r-dimensional vector spaces. We are interested in ***secant varieties*** of Segre varieties

$$\sigma_r = \overline{\bigcup \{\langle x_1, \dots, x_r \rangle \colon x_1, \dots, x_r \in \mathrm{Seg}\}} \subset \mathbb{P}(V_1 \otimes \dots \otimes V_d).$$

They arise naturally in the context of tensors and complexity theory.

The geometry of secant varieties is poorly understood. The ideal and the singular locus are known only for $r = 2$ and $r = d = 3$.

In [JJ], we introduce ***concise secant varieties***, which come with projective birational maps to the usual secant varieties $\rho \colon c\sigma_r \to \sigma_r$.

- The points of $c\sigma_r$ have a clear interpretation in the language of tensors, see §4.
- The local geometry of $c\sigma_r$ is controlled by **moduli spaces** such as Hilbert schemes of points, see §5.
- The singularities of $c\sigma_r$ are milder and better understood than for the usual secant varieties, see §2.
- The varieties $c\sigma_r$ admit natural torus actions, which allow one to study their topology, see §3.

2. Resolution of singularities of secant varieties

Using the smooth equivalences of §5, we translate known results on Hilbert and Quot schemes to obtain **resolutions of singularities**.

Theorem

- *For $r = 2, 3$, the variety $c\sigma_r$ is smooth and $\rho \colon c\sigma_r \to \sigma_r$ is a resolution of singularities of σ_r.*
- *For $r = 4$, the variety $c\sigma_4$ is normal, Gorenstein and has canonical singularities. Its singular locus has codimension 9. The blowup along the singular locus $\widetilde{c\sigma_4} := \mathrm{Bl}_{\mathrm{Sing}(c\sigma_4)} c\sigma_4$ is smooth and $\widetilde{\rho} \colon \widetilde{c\sigma_4} \to c\sigma_4 \to \sigma_4$ is a resolution of singularities.*

For $r = 5$, the only singularity of $c\sigma_5$ that is not fully understood corresponds to the singularity of $\mathrm{Hilb}_5(\mathbb{A}^4)$ at a fat point. It is open whether $\mathrm{Hilb}_5(\mathbb{A}^4)$ is Cohen-Macaulay there.

3. Topology of concise secant varieties

The concise secant varieties $c\sigma_r$ admit a natural **torus action**. For smooth varieties $c\sigma_2, c\sigma_3$ and $\widetilde{c\sigma_4}$ (see §2), this allows one to study their topology using the **Białynicki-Birula decomposition**.

Theorem

- *The varieties $c\sigma_2, c\sigma_3$ and $\widetilde{c\sigma_4}$ admit a torus action with isolated fixed points. In particular, they admit affine pavings.*
- *The motivic class $[c\sigma_2] \in K_0(\mathrm{Var})$ of $c\sigma_2$ is given by*

$$\frac{\mathbb{L}^4 - 1}{\mathbb{L} - 1} \cdot \left(\frac{(1+\mathbb{L})^{2(d-1)} + (1+\mathbb{L}^2)^{d-1}}{2} + (1+\mathbb{L})^{d-1} \cdot \frac{\mathbb{L}^{d-1} - \mathbb{L}}{\mathbb{L} - 1} \right).$$

  There is a similar polynomial expression for the motivic class of σ_2 calculated by using the resolution $\rho \colon c\sigma_2 \to \sigma_2$.
- *The Euler characteristic $\chi(c\sigma_3)$ of $c\sigma_3$ is given by*

$$9 \cdot \left(\binom{3^{d-1}}{3} + 2(d-1)3^{d-1}(3^{d-1} + 3(d-1) - 4) \right).$$

  Determining the motivic class of $c\sigma_3$ is work in progress.

4. Construction of concise secant varieties

Let $W_1, \dots, W_d$ be r-dimensional. A tensor in $W_1 \otimes \dots \otimes W_d$ is called ***concise*** if it does not lie in any proper subspace $W'_1 \otimes \dots \otimes W'_d$. A tensor has ***minimal border rank*** if it is concise and lies in σ_r.

Example

The ***multiplication tensor*** $\mu_R \in W_1 \otimes \dots \otimes W_d = R^\vee \otimes \dots \otimes R^\vee \otimes R$ of a length r algebra R is given by $\mu_R(a_1, \dots, a_{d-1}) = a_1 \cdot \ldots \cdot a_{d-1}$. The tensor μ_R has minimal border rank if and only if the algebra R is **smoothable**.

The variety $c\sigma_r$ is a **GIT quotient** of the variety

$$\{\widetilde{T} \in W_1 \otimes \dots \otimes W_d, (\varphi_i \colon W_i \to V_i)_{i=1,\dots,d} \colon \widetilde{T} \text{ minimal border rank}\}$$

via the natural action of $\mathrm{GL}(W_1) \times \dots \times \mathrm{GL}(W_d)$. The birational map $\rho \colon c\sigma_r \to \sigma_r$ is given by $[(\widetilde{T}, \varphi_1, \dots, \varphi_d)] \mapsto (\varphi_1 \otimes \dots \otimes \varphi_d)(\widetilde{T})$.

5. Smooth equivalences

Let $c\sigma_r^{\mathrm{fr}}$ be the locus of semistable points from the GIT quotient construction in §4. We have a smooth map $c\sigma_r^{\mathrm{fr}} \to c\sigma_r$ that sends a point to its orbit. There is also a smooth map $c\sigma_r^{\mathrm{fr}} \to \sigma_r \cap \{\text{concise}\}$ which forgets maps $\varphi_1, \dots, \varphi_d$. This gives a **smooth equivalence**

$$\begin{tikzcd} & c\sigma_r^{\mathrm{fr}} \arrow[dl, "\text{sm.}"'] \arrow[dr, "\text{sm.}"] & \\ c\sigma_r & & \sigma_r \cap \{\text{concise}\} \end{tikzcd}$$

which allows us to compare local singularities of these spaces. For example, a point in $c\sigma_r$ is smooth/normal/Gorenstein if and only if a corresponding point in $\sigma_r \cap \{\text{concise}\}$ has the same property.

Example

There is a smooth equivalence between the locus of multiplication tensors of algebras in $\sigma_r \cap \{\text{concise}\}$ and the smoothable component $\mathrm{Hilb}_r^{\mathrm{sm}}(\mathbb{A}^{r-1})$ of the **Hilbert scheme**, see §4.

For $r \le 3$, all minimal border rank tensors come from algebras. The Hilbert schemes of ≤ 3 points are smooth. Therefore, by the smooth equivalences above, the varieties $c\sigma_2, c\sigma_3$ are smooth.

6. Equations of secant varieties

The famous **Salmon problem** asks for the ideal defining

$$\sigma_4(\mathbb{P}^3 \times \mathbb{P}^3 \times \mathbb{P}^3) \subset \mathbb{P}^{63}.$$

Current approaches are based on direct computations. The theory of concise secant varieties gives a new approach to this problem. The variety $\widetilde{c\sigma_4}$ admits a natural torus action with isolated fixed points (see §3), so we can use **torus localization** to compute the Euler characteristic of line bundles $\mathcal{O}_{\widetilde{c\sigma_4}}(m)$. If Oeding's conjecture that σ_4 is arithmetically Cohen-Macaulay holds, then

$$\chi(\mathcal{O}_{\widetilde{c\sigma_4}}(m)) = h^0(\widetilde{c\sigma_4}, \mathcal{O}_{\widetilde{c\sigma_4}}(m)) = h^0(\sigma_4, \mathcal{O}_{\sigma_4}(m)),$$

so this would predict the number of equations in each degree.

Bibliography

[JJ] J. Jagiełła, J. Jelisiejew, *Unrestrictions and concise secant varieties*, preprint, 2026, arXiv:2604.24879.

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