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Some results on the Eisenbud - Green - Harris conjecture and the weak Lefschetz property

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The Eisenbud–Green–Harris (EGH) conjecture

Conjecture (Eisenbud–Green–Harris [1])

If $I \subseteq K[x_1, \ldots, x_n]$ is an homogeneous ideal containing a regular sequence $f_1, \ldots, f_n$, then there exists an ideal L that contains $x_1^{\deg(f_1)}, \ldots, x_n^{\deg(f_n)}$ and for all $d \in \mathbb{N}$ we have $\dim I_d = \dim L_d$.

The EGH conjecture aims to generalize classical theorems on Hilbert functions due to Macaulay, Kruskal, Katona, Clements, and Lindström. Many authors have worked on it for more than three decades, and most results can be found in the recent surveys [2, 3].

Defect δ ideals

A case of the EGH conjecture that has seen very little progress is when the polynomials in the regular sequence all have the same degree. The first person who made progress on this is Francisco [4]. He proved that the EGH conjecture holds in degrees $\leq d+1$, for an ideal generated by a regular sequence and one more polynomial of degree d. This prompted Hochster and Güntürkün to make the following definition.

Definition (Hochster–Güntürkün [5])

We say that a homogeneous ideal $I \subseteq K[x_1, \ldots, x_n]$ is a defect δ ideal, if I is equigenerated by $\delta + n$ polynomials and n of these polynomials form a regular sequence.

Hochster and Güntürkün proved that the EGH conjecture holds in degrees ≤ 3 for defect 2 ideals generated by quadratics. They also proved that the EGH conjecture holds in degrees ≤ 3 for defect 3 quadratic ideals in 5 variables. This allowed them to completely settle the EGH conjecture in 5 variables for ideals containing quadratic regular sequences.

Main theorem on the EGH conjecture

The following result says that the EGH conjecture holds for defect δ ideals generated in degree d, except for finitely many d.

Theorem (K 2026+)

Suppose that I is a defect δ ideal in a polynomial ring over a field of characteristic 0. There exists $D = D(\delta) \in \mathbb{N}$ such that if I is generated in degree $d \geq D$, then the EGH conjecture holds in degrees $\leq d+1$.

The weak Lefschetz property

Definition

For an Artinian homogeneous ideal $I \subseteq K[x_1, \ldots, x_n]$, a linear form ℓ, and $d \in \mathbb{N}$, we say that ℓ is a *weak Lefschetz element (WLE)* on $A = K[x_1, \ldots, x_n]/I$ in degree d, if the multiplication map by ℓ, $\times \ell_d : A_d \to A_{d+1}$, is injective or surjective. We say that ℓ is a WLE on A if ℓ is a WLE on A in degree d for all $d \in \mathbb{N}$. We say that A satisfies the *weak Lefschetz property (WLP)* if there exists a linear form ℓ that is a WLE on A.

An overview of Lefschetz property

The weak and strong (algebraic) Lefschetz properties are abstractions of the important Hard Lefschetz Theorem from complex geometry. Stanley in [6] proved the first result in the area, that every monomial complete intersection satisfies WLP and SLP.

A natural next step after Stanley is to prove that WLP holds for any complete intersection. There have been attempts in this direction [7, 8, 9]. The WLP has only been completely proved in 3 variables. In more than 3 variables all results are partial, mostly focusing on equigenerated complete intersections, and WLP only being satisfied in a some range of degrees.

Main theorem on the WLP

The following result says that the WLP holds for complete intersections in a range of degrees, after modding out by several linear forms. Note that the case $i = 0$ below is just the regular WLP, where if $i = 0$ then by $R/(I + (g_n, g_{n-1}, \ldots, g_{n-i+1}))$ we mean R/I.

Theorem (K 2026+)

Let $t \in \mathbb{N}$, and suppose $I \subseteq R = K[x_1, \ldots, x_n]$ is a complete intersection with degrees $e_1 \leq \cdots \leq e_n$ such that the consecutive differences $e_{i+1} - e_i$ are fixed. There exist linear forms $g_1, \ldots, g_n$ and an integer D such that if $e_1 \geq D$, then for all $i \in \{0, 1, \ldots, n-2\}$ and all $k \in \{0, 1, \ldots, t\}$ we have that g_{n-i} is a WLE in degree $e_1 - 1 + k$ on $R/(I + (g_n, g_{n-1}, \ldots, g_{n-i+1}))$.

Sketch of the proofs

We prove a much deeper result on generic initial ideals in order to obtain the previously mentioned theorems. It allows us to show that certain monomials must be in the generic initial ideal based on the length and degrees of a regular sequence. It also allows us to conclude much stronger theorems on Hilbert functions, Betti Numbers, and Lefschetz properties, which are not mentioned on the poster. The final paper will have the title *On the structure of generic initial spaces*.

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