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Shiji Lyu

Regular local rings over valuation rings

University of Illinois Chicago slyu@alumni.princeton.edu

Introduction

Valuation rings are integral domains whose ideals are totally ordered. Examples include fields k, the p-adic integers $\mathbf{Z}_p$, the power series ring $k[[T]]$ of one variable (those are Noetherian). In arithmetic geometry, we also have perfectoid valuation rings such as $\mathcal{O}_{\mathbf{C}_p}$ and $\mathbf{F}_p[[T^{1/p^\infty}]]$ (those are not Noetherian).

In this poster we discuss regularity over valuation rings. While the Noetherian case is relatively well-understood (covered by theorems applicable to all or most Noetherian rings), the non-Noetherian case seems to be severely lacking. We would like to extend results known in the Noetherian case to the general case, as well as to obtain applications back to Noetherian rings.

Setup

Let V be a valuation ring. We consider a local homomorphism of local rings $(V, \mathfrak{m}_V, k_V) \to (A, \mathfrak{m}_A, k_A)$ that is essentially finitely presented, *i.e.* $A = B_\mathfrak{q}$, where $B = V[T_1, \ldots, T_n]/(G_1, \ldots, G_m)$ and $\mathfrak{q} \in \operatorname{Spec} B$. Geometrically A corresponds to a singularity of an algebraic varity over V.

Even when V is not Noetherian, the ring A still possesses good ring-theoretic properties: it is *coherent*, which means every finitely presented A-module M has a free resolution by finitely generated free modules F_i: $\ldots \to F_2 \to F_1 \to F_0 \to M \to 0$. This means that many familiar homological methods can be applied, in particular it implies that M is flat if and only if $\operatorname{Tor}_1^A(k_A, M) = 0$.

Bertin (1972) defines A *regular* if all such resolutions can be made of finite length, *i.e.*, $F_i = 0$ for $i \gg 1$. He showed that a regular A must be a normal domain. In Knaf (2007) it was shown that in our setup, for a regular A, we can make $F_i = 0$ for $i > \dim A/\mathfrak{m}_V A + 1$. Note that $A/\mathfrak{m}_V A$ is Noetherian. This corresponds well to Serre's homological criterion of regularity of Noetherian local rings.

Main theorems

We extend many (other) known facts/criterions for regularity of Noetherian rings to our setup. We list two such results in their simpler form; full versions are rather long.

Theorem (L.; Noetherian case due to André)

Assume A contains a perfect field k. Then A is regular if and only if the cotangent complex $L_{A/k}$ is quasi-isomorphic to a flat module sitting in degree 0.

Theorem (L.; Noeth. case due to Kunz and Koh–Lee)

Assume A contains $\mathbf{F}_p$. Then A is regular if and only if all Frobenii F^e_*A are flat, if and only if some F^e_*A has finite tor dimension over A.

Here F^e_*A is the A-algebra given by the ring endomorphism $a \mapsto a^{p^e}$ on A. There is also a mixed-characteristic version of the result, parallel to work of Bhatt–Iyenger–Ma.

Other results include openness of regular locus of $\operatorname{Spec} B$ (which is not always true but is true when *e.g.* V has finite rank and B is flat), and construction and flatness of big Cohen–Macaulay algebras (suitably defined), from which it also follows that every regular A is a splinter.

Key ingredient

One key ingredient is Noetherian approximation, based on:

Theorem (follows from de Jong's alteration theorem)

Assume V is absolutely integrally closed (i.e. the fraction field of V is algebraically closed). Then $V = \bigcup_i R_i$ is a filtered union of regular local rings R_i essentially finitely presented over $\mathbf{Z}$ (in particular Noetherian).

Since B is defined by finitely many polynomials it is clear that we can find an i_0 and a finitely presented R_{i_0}-algebra B_{i_0} so that $B = B_{i_0} \otimes_{R_{i_0}} V$. Write $B_i = B_{i_0} \otimes_{R_{i_0}} R_i$. Grothendieck has studied such approximation situations much broadly (*i.e.* where V and R_i are arbitrary rings) in EGA IV$_3$, for example it was shown that if B is flat (resp. smooth) over V, then for all large i, so are B_i over V_i. In the non-flat case, specific to our setup, we proved

Theorem (L.)

For large i the ring B_i is tor-independent with R_j for all $j > i$, meaning that $\operatorname{Tor}^{R_i}_{>0}(B_i, R_j) = 0$, or equivalently that $B_j = B_i \otimes^L_{R_i} R_j$.

This is the key technical result for us to approximate various homological constructions, for example we have base change of relative cotangent complexes $L_{B_i/R_i} \otimes^L_{R_i} R_j = L_{B_j/R_j}$.

Application to Noetherian rings

We obtain a version of Kodaira's vanishing theorem over large enough residue characteristic. To the author's knowledge this is new even when restricted to the Noetherian case, *e.g.* $V = \mathbf{Z}_p$.

Theorem (L.)

Let $r, m, n, d \in \mathbf{Z}_{>0}$. There exists a constant $C = C(r, m, n)$ with the following property.
Assume $\operatorname{char} k_V = 0$ or $> C$. Let X be a closed subscheme of $\mathbf{P}^r_V$ defined by at most m homogeneous polynomials of degree at most n. Assume that X is flat over $\operatorname{Spec} V$ of relative dimension d, and assume that X is regular.
Let $\omega_X = \mathcal{E}xt^{r-d}_{\mathcal{O}_{\mathbf{P}^r_V}}(\mathcal{O}_X, \mathcal{O}_{\mathbf{P}^r_V}(-r-1))$. Then $H^j(X, \omega_X(1)) = 0$ for all $j > 0$.

When $\operatorname{char} k_V = 0$, we use Noetherian approximation to reduce to Kawamata–Viehweg vanishing for $\mathbf{Q}$-varieties. When $\operatorname{char} k_V > 0$, to obtain the bound, we assume by contradiction that we have counterexamples X_i over V_i with $\operatorname{char} k_{V_{i+1}} > \operatorname{char} k_{V_i}$. The boundedness assumption on number of equations and degrees then allow us to amalgamate X_i to get a scheme $\mathcal{X}$ over $\mathcal{V} := \prod_i V_i$. The ring $\mathcal{V}$ has distinguished maximal ideals $\mathfrak{M}_i$ so that $\mathcal{V}_{\mathfrak{M}_i} = V_i$; whereas for all other maximal ideals $\mathfrak{M}$ we have $\mathcal{V}_\mathfrak{M}$ is a valuation ring with residue characteristic 0. This allows us to get a contradiction.
Note: the localizations $\mathcal{V}_\mathfrak{M}$ are *ultraproducts* of V_i. We are doing logic (model theory) without actually doing logic :)

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