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Equivariant Koszul Cohomology of Canonical Curves

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The Koszul Complex

- Let $(X, \mathcal{O}_X)$ be a smooth, projective, irreducible scheme over an algebraically closed field k.
- Let L be a very ample line bundle on X.
- Let $V = H^0(L)$ and $S = \mathrm{Sym}(V) = \bigoplus_{q \geq 0} \mathrm{Sym}^q(V)$.

Definition

The **Koszul complex** associated to (X, L) is the complex of graded S-modules:

$$\cdots \xrightarrow{d} \bigwedge^{p+1} V \otimes H^0(L^{\otimes(q-1)}) \xrightarrow{d} \bigwedge^p V \otimes H^0(L^{\otimes q}) \xrightarrow{d} \bigwedge^{p-1} V \otimes H^0(L^{\otimes(q+1)}) \xrightarrow{d} \cdots$$

with differentials given by:

$$d_{p,q}(s_1 \wedge \cdots \wedge s_p \otimes f) = \sum_{i=1}^{p} (-1)^i s_1 \wedge \cdots \wedge \widehat{s_i} \wedge \cdots \wedge s_p \otimes (f s_i).$$

Its cohomology, denoted $K_{p,q}(X, L)$, is the (p,q)**-th Koszul cohomology** of (X, L).

The Equivariant Koszul Complex

- Let G be a finite group of order coprime to *char*(k) acting faithfully on the right on X.
- Let $L \in \mathrm{QCoh}_G(X)$ be a very ample line bundle on X, where $\mathrm{QCoh}_G(X)$ denotes the category of equivariant quasicoherent sheaves.
- $V = H^0(L)$ is a kG-module and $V \otimes \mathcal{O}_X \in \mathrm{QCoh}_G(X)$.
- The evaluation map $\mathrm{ev}_L : V \otimes \mathcal{O}_X \to L$
- Its kernel, $\mathcal{M}_L$, is the **Lazarsfeld bundle**, yielding the short exact sequence:

$$0 \longrightarrow \mathcal{M}_L \longrightarrow V \otimes \mathcal{O}_X \xrightarrow{\mathrm{ev}_L} L \longrightarrow 0$$

Theorem

$$[K_{p,q}(X, L)] = [H^0(\bigwedge^p \mathcal{M}_L \otimes L^{\otimes q})] - [H^0(\bigwedge^{p+1} V \otimes L^{\otimes(q-1)})] + [H^0(\bigwedge^{p+1} \mathcal{M}_L \otimes L^{\otimes(q-1)})].$$

Why this construction is equivariant.

- **Lifting.** Functors naturally compatible with the G-action lift to the corresponding categories of G-objects.
- **Operations.** The functors $\otimes$, $\bigwedge^p$, H^0, and $W \mapsto W \otimes \mathcal{O}_X$ preserve G-structures.
- **Contractions.** For an exact sequence $0 \to \mathcal{H} \to \mathcal{F} \to \mathcal{Q} \to 0$ of G-vector bundles with $\mathrm{rk}\, \mathcal{Q} = 1$, the induced sequence

$$0 \to \bigwedge^p \mathcal{H} \to \bigwedge^p \mathcal{F} \to \bigwedge^{p-1} \mathcal{H} \otimes \mathcal{Q} \to 0$$

is exact and equivariant.

Canonical Curves

Let $(X, \mathcal{O}_X)$ be a smooth, projective, irreducible curve of genus $g \geq 4$. Let $V = H^0(\Omega_X)$ and $\mathcal{M}_{\Omega_X} = \ker \mathrm{ev}_{\Omega_X}$.

Corollary

For $1 \leq p \leq g-3$*:*

$$[K_{p,1}(X, \Omega_X)] - [K_{p-1,2}(X, \Omega_X)] = [\bigwedge^p V \otimes V] - [\bigwedge^{p+1} V] - [H^0(\bigwedge^{p-1} \mathcal{M}_{\Omega_X} \otimes \Omega_X^{\otimes 2})].$$

The first two terms form the Schur functor

$$\mathbb{S}_p(V) := \ker \left(\bigwedge^p V \otimes V \longrightarrow \bigwedge^{p+1} V \right).$$

Hence the problem becomes

$$[K_{p,1}] - [K_{p-1,2}] = [\mathbb{S}_p(V)] - [H^0(X, \mathcal{E}_{p-1})], \qquad \mathcal{E}_r := \bigwedge^r \mathcal{M}_{\Omega_X} \otimes \Omega_X^{\otimes 2}.$$

Two computations.

- **Schur side:** compute $[\mathbb{S}_p(V)]$ from the character of $V = H^0(\Omega_X)$.
- **Vector-bundle side:** in the relevant range, $H^1(X, \mathcal{E}_r) = 0$, so $[H^0(X, \mathcal{E}_r)] = \chi_G(\mathcal{E}_r)$. Ellingsrud–Lønsted equivariant Riemann–Roch computes $\chi_G(\mathcal{E}_r)$ from rank, degree and ramification data.

These two terms are computed explicitly in the boxes on the right.

Explicit Formulas for Canonical Curves

- **Quotient data:** $\pi : X \to Y = X/G$; write g_X, g_Y for the genera of X, Y.
- **Ramification data:** For $Q \in Y$, choose $P \in \pi^{-1}(Q)$. Then $G_P = \mathrm{Stab}_G(P)$ is cyclic of order e_Q.
- **Characters:** χ_P is the fundamental character of G_P on $\mathfrak{m}_P/\mathfrak{m}_P^2$, and $\mathrm{Irr}(G_P) = \{\chi_P^d : 0 \leq d < e_Q\}$.
- **Local multiplicities:** $n_{d,Q,I} := \langle \chi_P^d, \mathrm{Res}_{G_P}^G I \rangle_{G_P}$.

Theorem

$$\begin{aligned}
&[H^0(\bigwedge^p \mathcal{M}_{\Omega_X} \otimes \Omega_X^{\otimes 2})] \\
&= \binom{g_X - 1}{p} \left(\frac{2(g_X - 1 - p)}{|G|} + g_Y - 1 \right) [kG] \\
&+ \sum_{I \in \mathrm{Irr}(G)} \sum_{Q \in Y} \sum_{i=0}^{p} (-1)^i \sum_{d=0}^{e_Q - 1} \\
&\langle \chi_P^d, \bigwedge^{p-i} H^0(\Omega_X) \rangle \sum_{k=0}^{e_Q - 1} \left\{ \frac{d + i + 1 - k}{e_Q} \right\} n_{k,Q,I} [I].
\end{aligned}$$

Schur Functors

The virtual representation $[\bigwedge^p V \otimes V] - [\bigwedge^{p+1} V]$ is the kernel of the canonical epimorphism $\bigwedge^p V \otimes V \to \bigwedge^{p+1} V$. This is isomorphic to the *Schur functor*:

$$\mathbb{S}_p(V) := \mathbb{S}_{(2,1,\ldots,1)}(V) = \mathbb{S}_{(2,1^{p-2})}(V)$$

Theorem

For $g \in G$ *and* $I \in \mathrm{Irr}(G)$*, let* $\xi_{I,1}, \ldots, \xi_{I,\dim I}$ *be the eigenvalues of* g *on* I*. Let the multiplicity of* I *in* V *be:*

$$m_I = (g_Y - 1) \dim I + \delta_{I, I_{\mathrm{triv}}} + \sum_{Q \in Y} \sum_{d=0}^{e_Q - 1} \left\langle \frac{-d}{e_Q} \right\rangle n_{d,Q,I}$$

Then $[\mathbb{S}_p(V)]$ *is the coefficient of* t^p *in:*

$$\boxed{\frac{1}{t} \left(\frac{1}{|G|} \left(t \sum_I m_I [I] - [I_{\mathrm{triv}}] \right) \sum_I \sum_{g \in G} \prod_J \prod_{j=1}^{\dim J} (1 + t \cdot \xi_{J,j}(g))^{m_J} \overline{\chi_I(g)} [I] + [I_{\mathrm{triv}}] \right) \in \mathrm{Rep}_k(G).}$$

Totally Ramified Trigonal Curves

Let $s \in \mathbb{Z}_{>0}$ with $3 \mid s$, and distinct $a_i \in \mathbb{C}$. The curve $X : y^3 = \prod_{i=1}^{s} (x - a_i)$ is a trigonal curve of genus $g = s - 2$.

Proposition

For $1 \leq p \leq g - 2$*:*

$$[\mathbb{S}_p(V)] \cong \bigoplus_{j=0}^{p+1} \left[(p-j) \binom{\frac{s}{3} - 1}{j} \binom{\frac{2s}{3}}{p+1-j} + \left(\frac{s}{3} - 1\right) \binom{\frac{s}{3} - 1}{j - 1} \binom{\frac{2s}{3} - 1}{p + 1 - j} \right] \chi_{(p+j+1) \bmod 3}$$

$$\begin{aligned}
[H^0(\bigwedge^p \mathcal{M}_{\Omega_X} \otimes \Omega_X^{\otimes 2})] &= \left[\frac{1}{3} \binom{s-3}{p} (2s - 2p - 9) \right] \sum_{r=0}^{2} \chi_r \\
&+ \sum_{i=0}^{p} \sum_{r=0}^{2} \sum_{d=0}^{2} \sum_{\substack{0 \leq j \leq p-i \\ (p-i+j) \equiv d\ (3)}} (-1)^i s \binom{2s/3 - 1}{p - i - j} \binom{s/3 - 1}{j} \left\{ \frac{d + i + 1 - r}{3} \right\} \chi_r
\end{aligned}$$

p	$[\mathbb{S}_p(V)]$	$[H^0(\bigwedge^{p-1} \mathcal{M} \otimes \Omega^{\otimes 2})]$	$[K_{p,1}] - [K_{p-1,2}]$
1	$10\chi_0 + 3\chi_1 + 15\chi_2$	$6\chi_0 + 3\chi_1 + 9\chi_2$	$\mathbf{4}\chi_0 + \mathbf{6}\chi_2$
2	$42\chi_0 + 50\chi_1 + 20\chi_2$	$38\chi_0 + 38\chi_1 + 20\chi_2$	$\mathbf{4}\chi_0 + \mathbf{12}\chi_1$
3	$55\chi_0 + 55\chi_1 + 100\chi_2$	$55\chi_0 + 64\chi_1 + 91\chi_2$	$-\mathbf{9}\chi_1 + \mathbf{9}\chi_2$
4	$100\chi_0 + 80\chi_1 + 44\chi_2$	$104\chi_0 + 80\chi_1 + 56\chi_2$	$-\mathbf{4}\chi_0 - \mathbf{12}\chi_2$
5	$25\chi_0 + 50\chi_1 + 65\chi_2$	$29\chi_0 + 56\chi_1 + 65\chi_2$	$-\mathbf{4}\chi_0 - \mathbf{6}\chi_1$

Table 1: Comparison of Schur Functor and Vector Bundle Cohomology for $s = 9$.

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