# Commutative Algebra and Algebraic Geometry in Milan

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# Cohomology on the Incidence Correspondence

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## Incidence Correspondence

Let $k$ an algebraically closed field. The **incidence correspondence** $X$ is the partial flag variety parametrizing subspaces $V_1 \subseteq V_{n-1}$ of dimension 1 and $n-1$, respectively. So we have

$$X = \{(x, H) \mid x \in H\} \subseteq \mathbb{P} \times \mathbb{P}^\vee,$$

where $\mathbb{P} = \mathbb{P}(k^n) = \mathbb{P}^{n-1}$.

### Line Bundles on X

Each line bundle on $X$ is the restriction of a line bundle on $\mathbb{P} \times \mathbb{P}^\vee$, hence of the form $\mathcal{O}_X(a,b) = \mathcal{O}_{\mathbb{P}\times\mathbb{P}^\vee}(a,b)|_X$ for $(a,b) \in \mathbb{Z}^2$.

### Goal

Compute the **dimension** and the **character** of the sheaf cohomology of line bundles on $X$ in positive characteristic.

- The cohomology groups of $\mathcal{O}_X(a,b)$ are finite-dimensional representations of $\mathrm{GL}_n$.
- The *character* records the eigenspace decomposition of $W$ relative to the action of $(k^\times)^n$:

$$[W] = \sum \dim(W_{(i_1,\dots,i_n)})\, z_1^{i_1} \cdots z_n^{i_n}.$$

- $[W]$ is a symmetric Laurent polynomial in $\mathbb{Z}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$.

In $\mathrm{char}(k) = 0$, the cohomology of line bundles over flag varieties is known by the Borel–Weil–Bott theorem.
In $\mathrm{char}(k) = p > 0$, it is an open question.

### Ex: $n = 4$, $\mathcal{L} = \mathcal{O}_X(5, -7)$

| | $\dim H^2(X,\mathcal{L})$ | $\dim H^3(X,\mathcal{L})$ |
|---|---|---|
| $\mathrm{ch}(k) = 0$ | 105 | 0 |
| $\mathrm{ch}(k) = 2$ | 136 | 31 |
| $\mathrm{ch}(k) = 3$ | 120 | 15 |

## Previous Work

- **Griffith 1980**
  - vanishing behaviour of cohomology for $n = 3$.
- **Donkin 2007, Liu 2021**
  - recursive formula for $n = 3$.
- **Gao–Raicu 2024**
  - reformulate the recursive formula ($n = 3$);
  - closed formula for $n = 3$ and $p = 2$;
  - characterization of vanishing for all $n$.
- **O'Dorney 2025**
  - closed formula for $n = 3$.

### Equivalent Reformulation

When $a \geq 0$, $b \leq -n$, by *[Gao–Raicu]*:

$$h^i(D^d\mathcal{R}(e)) \cdot z_1 \cdots z_n = [H^{i+n-2}(X, \mathcal{O}_X(e+1, -d-n+1))]$$

where $\Omega$ is the cotangent sheaf on $\mathbb{P}^{n-1}$, $\mathcal{R} = \Omega(1)$, $D^d\mathcal{R}$ its $d$-th divided power, and $h^i(D^d\mathcal{R}(e)) = [H^i(\mathbb{P}^{n-1}, D^d\mathcal{R}(e))]$.

**Key reductions:**

- In all other cases $h^i(D^d\mathcal{R}(e))$ is independent of $p$.
- $h^i(D^d\mathcal{R}(e)) = h^{1-i}(D^{e+1}\mathcal{R}(d-1))$, so we may assume $e \geq d-1$.
- If $d < p$, the result is the same as in characteristic 0.

## Main Result

### Recursive formula for any $n$

**Theorem [KMRR].**

If $e \geq d-1$, then for $i = 0, 1$:

$$h^i(D^d\mathcal{R}(e)) = \sum_{a=0}^{\lfloor \frac{d}{p} \rfloor} \sum_{b=-1}^{\lfloor \frac{d+e}{p} \rfloor} \Phi_{d-ap,\, e-bp} \cdot F^p\Big(h^i(D^a\mathcal{R}(b))\Big)$$

where

- $\Phi_{d,e} = \sum_{j\geq 0} (h'_{e+jp} h'_{d-jp} - h'_{e+1+jp} h'_{d-1-jp})$
- $F^p : z_i \mapsto z_i^p$ is the Frobenius
- $h'_k$ is the $p$-truncated complete symmetric polynomial

### Closed-form in char. 2

Currently working on a closed formula for any $n$.

## Application: the WLP

An Artinian algebra $A$ has the **Weak Lefschetz Property (WLP)** if there exists $\ell \in A_1$ such that $\times \ell : A_i \to A_{i+1}$ has maximal rank for all $i$.

### How is this connected?

We show that for a monomial **Complete Intersection (CI)** in $\mathrm{char}(k) = p$:

$A = \dfrac{k[x_1, \dots, x_n]}{(x_1^{a_1}, \dots, x_n^{a_n})}$ of socle degree $s = \sum(a_i - 1)$ does **not** have the WLP

**if and only if**

$z_1^{a_1-1} \cdots z_n^{a_n-1}$ appears in the character of $H^1(\mathbb{P}^{n-1}, D^d\mathcal{R}(e))$ for $d+e = s$, $d-1 \leq e \leq d$

## WLP in characteristic 2

We define

$$\theta_q(x) = \begin{cases} r & r \leq q-1, \\ 2q-2-r & q-1 \leq r \leq 2q-2, \\ -\infty & r = 2q-1, \end{cases}$$

$r$ being the remainder of $x$ modulo $2q$.

### WLP for monomial CIs in char. 2

**Theorem [KMRR].**

$k[x_1, \dots, x_n]/(x_1^{a_1}, \dots, x_n^{a_n})$ has the WLP if and only if

$$\sum_{1 \leq i \leq n} \theta_q(a_i - 1) \leq 2q - 2$$

for all $q = 2^k$ satisfying $\frac{(a_1-1)\oplus\cdots\oplus(a_n-1)}{2} < q \leq 2(a_n - 1)$, where $\oplus$ is the Nim sum.

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