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Waring rank of binomials with apolar ideal as a complete intersection

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Introduction

Let $S = \mathbb{C}[x_0, \dots, x_n] = \bigoplus_{i \geq 0} S_i$ be the standard graded polynomial ring over $\mathbb{C}$, and $F \in S$ be a homogeneous polynomial of degree d. It is well-known that there exist $L_1, \dots, L_r \in S_1$ such that

$$F = \sum_{i=1}^{r} \alpha_i L_i^d,$$

for some $\alpha_i \in \mathbb{C}$, $1 \leq i \leq r$. The Waring rank of F, denoted by $\mathrm{Rk}\, F$, is

$$\mathrm{Rk}(F) := \min\{r : F = \sum_{i=1}^{r} \alpha_i L_i^d \text{ for some } L_i \in S_1\}.$$

Question: Given a form F of degree d, what is $\mathrm{Rk}(F)$?

What is known?

- In 1995 J. Alexander and A. Hirschowitz [1] determined the Waring rank of a generic form.
- The Waring rank of monomials is computed by E. Carlini, M. V. Catalisano, and A. V. Geramita in 2012 [3], namely,

$$\mathrm{Rk}(x_0^{a_0} x_1^{a_1} \dots x_n^{a_n}) = \prod_{i=1}^{n} (a_i + 1) \text{ where } 1 \leq a_0 \leq a_1 \leq \dots \leq a_n$$

- In 2021 L. Brustenga and S. K. Masuti [2] computed the Waring rank of binary binomials.

Background

- Let $T = \mathbb{C}[X_0, \dots, X_n]$ be the ring of differential operators on S. The apolarity action of T on S is defined on the monomials as

$$X_0^{a_0} \cdots X_n^{a_n} \circ (x_0^{b_0} \cdots x_n^{b_n}) = \frac{\partial^{a_0}}{\partial x_0^{a_0}} \cdots \frac{\partial^{a_n}}{\partial x_n^{a_n}} (x_0^{b_0} \cdots x_n^{b_n}),$$

  and then extended linearly on S.
- Given a homogeneous form $F \in S_d$, we define the apolar ideal of F as

$$F^\perp = \{g \in T \mid g \circ F = 0\} \subseteq T.$$

- In this project we give an explicit formula for the Waring rank of a binomial F such that $T/F^\perp$ is a complete intersection.

Main theorem

Let $F = x_0^{a_0} x_1^{a_1} \cdots x_n^{a_n} (x_0^{b_0} - x_1^{b_1} \cdots x_n^{b_n})$. Write $F = F_1 F_2$ where $F_1 = x_0^{a_0} x_1^{a_1} \cdots x_k^{a_k} (x_0^{b_0} - x_1^{b_1} \cdots x_k^{b_k})$ a binomial with $b_i \neq 0$ for $1 \leq i \leq k$, $b_{k+1} = \cdots = b_n = 0$, and $F_2 = x_{k+1}^{a_{k+1}} \cdots x_n^{a_n}$ for some $1 \leq k \leq n$, and $a_{k+1} \leq \cdots \leq a_n$. Let $\lambda = \min(\{a_i + b_i + 1 \mid 1 \leq i \leq k\} \cup \{a_{k+1} + 1\})$. Assume that $k \geq 2$ and $T/F^\perp$ is a complete intersection. Then

$$\mathrm{Rk}(F) = \begin{cases} \frac{(a_0+1)}{\lambda} \prod_{i=1}^{k} (a_i + b_i + 1) \prod_{i=k+1}^{n} (a_i + 1) & \text{if } \lambda \leq a_0 + 1 \\ \prod_{i=1}^{k} (a_i + b_i + 1) \prod_{i=k+1}^{n} (a_i + 1) & \text{otherwise} \end{cases}$$

We have an explicit formula for the Waring rank of F if $k = 1$ as well.

Sketch of the proof

1. In [4, 5] the authors have characterized a binomial F such that $T/F^\perp$ is a complete intersection.

Sketch of the proof continued

- In fact, they showed that the apolar ideal $F^\perp$ of such a binomial F is a *quasi-monomial complete intersection* in "most" of the cases, that is, an ideal of the form

$$F^\perp = (X_0^{a_0} + G, X_1^{a_1}, X_2^{a_2}, \dots, X_n^{a_n}) \tag{1}$$

  where G is a homogeneous form in T of degree a_0 with $\deg_{X_0} G < a_0$.
- If F is as in (1) with $a_1 \leq a_2 \leq \dots \leq a_n$. Then

$$\mathrm{Rk}(F) = \begin{cases} a_0 \prod_{j=2}^{n} a_j & \text{if } a_1 \leq a_0 \text{ or} \\ & a_1 > a_0 \text{ and } (X_0^{a_0} + G) \text{ does not have a multiple factor.} \\ \prod_{j=1}^{n} a_j & \text{if } a_1 > a_0 \text{ and } (X_0^{a_0} + G) \text{ has a multiple factor.} \end{cases}$$

- We use e-compatibility method to get a lower bound on the $\mathrm{Rk}(F)$, and apolarity lemma to get an upper bound on $\mathrm{Rk}(F)$.

Examples

- Let $F = x_0^3 x_1 x_2^2 x_3^3 (x_0^4 - x_1^2 x_2^2) \in \mathbb{C}[x_0, x_1, x_2, x_3]$. Then $\mathrm{Rk}(F) = 80$.
- Let $F = x_0^2 x_1^3 x_2^3 x_3^4 (x_0^6 - x_1^4 x_2 x_3) \in \mathbb{C}[x_0, x_1, x_2, x_3]$. Then $\mathrm{Rk}(F) = 240$.

What is next?

- What is the Waring rank of any binomial form F in $\mathbb{C}[x, y, z]$.
- the Variety of Sum of Powers is

$$VSP(F, r) = \{\mathbb{X} \in \mathcal{H}ilb_r(\mathbb{P}^n) : I(\mathbb{X}) \subseteq F^\perp\},$$

  where $\mathcal{H}ilb_r(\mathbb{P}^n)$ denotes the Hilbert scheme of length r zero-dimensional subschemes of $\mathbb{P}^n$.
- Decribe the Variety of Sum of Powers $VSP(F, r)$ of F such that F is a binomial with $T/F^\perp$ a CI and $r = \mathrm{Rk}(F)$.

Bibliography

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