Commutative Algebra and Algebraic Geometry in Milan

Politecnico di Milano, June 29–July 3, 2026

Ryo Ishizuka   Sora Miyashita

The trace of the exterior powers of Kähler differentials

Osaka University   u804642k@ecs.osaka-u.ac.jp

Introduction

The content of this poster is based on **[IM26]**.
For a commutative ring S and an S-module M, the *trace* of M is

$$\mathrm{tr}_S(M) := \sum_{f \in \mathrm{Hom}_S(M,S)} f(M).$$

For a C–M local or graded ring, the trace of the canonical module measures how far the ring is from being Gorenstein (**[HHS19]**). We study the analogous problem for the modules of differentials; for background on trace ideals, see also **[Lin17]**.

Let A be essentially of finite type over a perfect field $\mathbf{k}$. We focus on $\Omega^i_{A/\mathbf{k}} := \bigwedge^i_A \Omega^1_{A/\mathbf{k}}$, especially on the top exterior power $\Omega^{\dim A}_{A/\mathbf{k}}$.

Definition. We call $\mathrm{tr}_A(\Omega^{\dim A}_{A/\mathbf{k}})$ the *top differential trace*.

Main goals:

- characterize polynomial rank via traces of Ω^i;
- detect regularity from the top differential trace;
- study the new notion of *nearly regular rings*.

Basic setup

Let $R = \bigoplus_{n \geq 0} R_n$ be a Noetherian graded ring with $R_0 = \mathbf{k}$ a field, and let A be a local $\mathbf{k}$-algebra essentially of finite type.

Polynomial rank ([Cos79]). We denote by $\mathrm{p.rk}(R)$ the largest integer $n \geq 0$ such that $R \cong T[x_1, \ldots, x_n]$ as graded rings for some graded subring $T \subseteq R$, where each x_i is homogeneous.

We denote by $\mathrm{fp.rk}(\widehat{A})$ the largest integer $n \geq 0$ such that $\widehat{A} \cong T[\![x_1, \ldots, x_n]\!]$ for some complete local $\mathbf{k}$-subalgebra $T \subseteq \widehat{A}$.

MAIN THEOREM ([IM26])

Assume that $\mathrm{char}\, \mathbf{k} = 0$.

(1) Rank detection. For every $i \geq 0$,

$$\mathrm{tr}_R(\Omega^{\dim R - i}_{R/\mathbf{k}}) = R \iff \mathrm{p.rk}(R) \geq \dim R - i;$$

$$\mathrm{tr}_{\widehat{A}}(\widehat{\Omega}^{\dim A - i}_{\widehat{A}/\mathbf{k}}) = \widehat{A} \iff \mathrm{fp.rk}(\widehat{A}) \geq \dim A - i.$$

Now let S denote either R or A.

(2) Regularity criterion. If S is reduced, then

$$\mathrm{tr}_S(\Omega^{\dim S}_{S/\mathbf{k}}) = S \iff S \text{ is regular}.$$

(3) Singular locus. If S is reduced and equidimensional, then

$$V\Big(\mathrm{tr}_S(\Omega^{\dim S}_{S/\mathbf{k}})\Big) = \mathrm{Sing}(S).$$

Sketch of the proof

Step 1: If $\mathrm{tr}_S(\Omega^1_{S/S_0}) = S$, then Ω^1_{S/S_0} has a free summand. Hence there exist an S_0-derivation $D : S \to S$ and an element $t \in S$ such that $D(t) = 1$; in other words, D admits a slice.

Step 2: If $\mathrm{char}\, \mathbf{k} = 0$, the slice theorem (see **[Fre17]**) gives $S \cong (\ker D)[t]$ in the graded case, and $\widehat{S} \cong (\ker D)[\![t]\!]$ in the local case.

Step 3: If $\mathrm{tr}_S(\Omega^{\dim S}_{S/\mathbf{k}}) = S$, then $\Omega^{\dim S}_{S/\mathbf{k}}$ has a free summand. For reduced rings, this forces S to be regular. Thus, unlike the classical Jacobian criterion, a single free summand in $\Omega^{\dim S}_{S/\mathbf{k}}$ is enough.

Step 4: With an equidimensionality assumption, a localization argument shows $V(\mathrm{tr}_S(\Omega^{\dim S}_{S/\mathbf{k}})) = \mathrm{Sing}(S)$.

Nearly regular rings

Let $\mathbf{k}$ be a field, and assume for simplicity that $\mathrm{char}(\mathbf{k}) = 0$.
Let S be either a Noetherian $\mathbb{N}$-graded $\mathbf{k}$-algebra with $S_0 = \mathbf{k}$, or a local $\mathbf{k}$-algebra essentially of finite type over $\mathbf{k}$.

Following our result and **[HHS19]**, we say that S is *nearly regular* if $\mathrm{tr}_S(\Omega^{\dim S}_{S/\mathbf{k}}) \supseteq \mathfrak{m}_S$, where $\mathfrak{m}_S$ is the irrelevant ideal of S.

Basic features.

- In the reduced setting, regular rings are nearly regular.
- For **1-dimensional Noetherian graded rings over a field**, nearly regularity is automatic.
- In higher dimension, the condition is much more restrictive...

If S is moreover **reduced and equidimensional**, then nearly regularity implies that S has an isolated singularity.

Applications and examples

The following applications and examples come from **[IM26]**.

Tensor products. Let A and B be reduced equidimensional graded $\mathbf{k}$-algebras, and set $R := A \otimes_{\mathbf{k}} B$. Then we have

$$\mathrm{tr}_R(\Omega^{\dim R}_{R/\mathbf{k}}) = \mathrm{tr}_A(\Omega^{\dim A}_{A/\mathbf{k}})R \cdot \mathrm{tr}_B(\Omega^{\dim B}_{B/\mathbf{k}})R.$$

In this setting, R is nearly regular if and only if it is regular.

Fiber products. Let A and B be reduced equidimensional graded $\mathbf{k}$-algebras of positive dimension with $A_0 = B_0 = \mathbf{k}$, and set $R := A \times_{\mathbf{k}} B$ with respect to the natural projections to $\mathbf{k}$. Then R is nearly regular $\iff$ both A and B are nearly regular and $\dim A = \dim B$.

Stanley–Reisner rings. Let Δ be a simplicial complex with pure connected components $\Delta_1, \ldots, \Delta_n$, and set $d_i := \dim \Delta_i$. Then:
$\mathbf{k}[\Delta]$ is nearly regular $\iff$ each Δ_i is a simplex and $d_1 = \cdots = d_n$.

Conclusions

- Trace ideals of exterior powers of differentials detect hidden polynomial and formal variables.
- For reduced equidimensional rings, the top differential trace gives a new computable ideal defining the singular locus.
- This complements other known descriptions of $\mathrm{Sing}(S)$, such as the Fitting ideal of Ω^1 and the cohomological annihilator of Iyengar–Takahashi (**[IT16]**).
- Nearly regular rings provide a natural regular analogue of nearly Gorenstein rings (**[HHS19]**).
- Tensor products, fiber products, and Stanley–Reisner rings illustrate applications of the theory.

Bibliography

[Cos79] C. Costa, *Polynomially complete algebras and independent sets*, Colloq. Math. **40** (1979), 101–106.

[Fre17] G. Freudenburg, *Algebraic Theory of Locally Nilpotent Derivations*, 2nd ed., Springer, 2017.

[HHS19] J. Herzog, T. Hibi, D. I. Stamate, *The trace of the canonical module*, Israel J. Math. **233** (2019), 133–165.

[IT16] S. B. Iyengar, R. Takahashi, *Annihilation of cohomology and strong generation of module categories*, IMRN **2016**, no. 2, 499–535.

[Lin17] H. Lindo, *Trace ideals and centers of endomorphism rings of modules over commutative rings*, J. Algebra **482** (2017), 102–130.

[IM26] R. Ishizuka, S. Miyashita, *Trace ideals of exterior powers of the module of differentials*, preprint, 2026.

Sponsored by: MUR PRIN   PRIN 2022 (MUR grant 2022K48YYP)
Unirationality, Hilbert schemes and Singularities
Foundation Compositio Mathematica