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# Linkage and the Watanabe–Yoshida Conjecture

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## Abstract

**The Watanabe–Yoshida conjecture** posits that the Hilbert–Kunz multiplicity of any singularity is bounded below by that of the $A_1$ singularity of the same dimension. While this has been verified in some cases, the general result remains open. Using a structure theorem by Huneke and Ulrich, we settle the conjecture for licci ideals. Our method of proof opens the door to approaching the conjecture through **linkage and essential deformation.**


## Watanabe–Yoshida Conjecture

**The Hilbert–Kunz multiplicity.** Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$ and dimension $d$. Write $\mathfrak{m}^{[p^e]} := (x^{p^e} : x \in \mathfrak{m})$. Then,

$$e_{\mathrm{HK}}(R) := \lim_{e\to\infty} \frac{\operatorname{length}\left(R/\mathfrak{m}^{[p^e]}\right)}{p^{de}} \in \mathbb{R}_{\geq 1}.$$

It **detects singularity** and tight closure. **It need not be an integer**, and $e_{\mathrm{HK}}(R) \leq e(R)$, where $e(R)$ is the Hilbert-Samuel multiplicity.

**The Watanabe–Yoshida conjecture (2005).** *Let $(R, \mathfrak{m}, k)$ be an equidimensional non-regular local ring of characteristic p and dimension d, and*

$$R_{p,d} := \frac{k[[x_0, \ldots, x_d]]}{(x_0^2 + \cdots + x_d^2)} \quad (A_1 \text{ singularity})$$

*Then $e_{\mathrm{HK}}(R) \geq e_{\mathrm{HK}}(R_{p,d})$, and equality holds if and only if $\hat{R} \cong R_{p,d}$.*

The conjecture has been settled in some cases:

1. *Low-dimensional cases:* up to dimension 7 (1, 2, 4).
2. *Complete intersections* (3, 4).
3. *Low-multiplicity cases:* up to (Hilbert–Samuel) multiplicity 5. Also, CM minimal multiplicity (2).

## Linkage Background

Let $I$ and $J \subset R$ be proper Cohen-Macaulay ideals.

**Linkage:** $I$ and $J$ are ***linked*** if there exists a complete intersection ideal $C$ such that $C :_R I = J$ and $C :_R J = I$, and we write $I \sim_C J$. We say $I$ and $J$ are in the same ***linkage class*** if $I = I_0 \sim I_1 \sim \cdots \sim I_n = J$.

**Licci ideals:** $I$ is ***licci*** if it is in the ***li**nkage **c**lass of a **c**omplete **i**ntersection*. This is the "nicest" linkage class.

**Examples:** CM ideals of grade 2, Gorenstein ideals of grade 3.

## Essential Deformations and Universal Links

We say that $(S, J)$ is an **essential deformation** of $(R, I)$ if $(R, I)$ can be obtained after a sequence of *localisations*, *regular sections*, and *extensions* of the residue field. Note: $e_{\mathrm{HK}}(S/J) \leq e_{\mathrm{HK}}(R/I)$.

**Universal linkage.** The $j$-th universal link is an ideal $L^j(I) \subset R(X)$ such that $(R(X), L^j(I))$ *is an essential deformation of* $(R, J)$ where $J$ is any $j$-th link of $I$. Since $I$ is linked to itself in $2n$ steps, then $(R(X), L^{2n}(I))$ *is also an essential deformation of* $(R, I)$.

$$\begin{tikzcd}
I = I_0 \arrow[drrr, dotted, "\text{ess def}"'] & \sim & I_1 & \sim & \ldots & \sim & I_{2n} = J \arrow[d, dotted, "\text{ess def}"] \\
 & & & & & & L^{2n}(I)
\end{tikzcd}$$

## Main Result

Say $I \subset R$ be a **licci ideal** so that $R/I$ is a non-regular local ring of dimension $d$ and characteristic $p > 0$. Then $e_{\mathrm{HK}}(R/I) > e_{\mathrm{HK}}(R_{p,d})$, that is, **the Watanabe–Yoshida conjecture holds for licci ideals**.

## Proof Outline

Our proof relies on the following main results:

**Structure theorem of Licci ideals (Huneke–Ulrich, 1987).** Let $(R, \mathfrak{m}, k)$ be a regular local ring and let $I \subset R$ be a licci ideal which is not a complete intersection. Then:

$$\begin{tikzcd}
 & (R, I) \arrow[dl, dotted, "\text{ess def}"'] \arrow[dr, dotted, "\text{ess def}"] & \\
(P[X_{5\times5}], \mathrm{Pf}_4(X)) & \text{or} & (P[X_{3\times2}], I_2(X))
\end{tikzcd}$$

**Aberbach and Enescu's lower bounds (2012).** The Hilbert–Kunz multiplicity of those two rings is large enough, and so

$$e_{\mathrm{HK}}(R/I) > e_{\mathrm{HK}}(R_{p,d}).$$

The result thus follows.

## Even Linkage Class of Minimal Multiplicity

Say $I$ is in the even linkage class of an ideal $J$ such that $R/J$ is non-regular and of minimal multiplicity. Assume that $L^{2n}(I) \subset \mathfrak{m}^2$.

Then $S/L^{2n}(I)$ is non-regular and of minimal multiplicity.

Thus $e_{\mathrm{HK}}(R/I) \geq e_{\mathrm{HK}}(S/L^{2n}(I)) > e_{\mathrm{HK}}(R_{p,d})$.

## Minimal Multiplicity Defect

In order to generalize the statement, consider the following obstruction to minimal multiplicity:

$$D(R) := e(R) - \operatorname{embdim}(R) + \dim R$$

**Natural question:** Does this invariant decrease under essential deformations? Not true in general.

**Example:** Let $R = k[x, y, z]/(x^2, xy, xz - y^2)$. $D(R) = 1$, $D(R_{(x,y)}) = 2$.

However, the answer is unclear for Cohen-Macaulay *domains*.

## Conclusions and Further Questions

- The Watanabe-Yoshida Conjecture is true for licci ideals!
- Does the minimal multiplicity defect of a Cohen-Macaulay domain decrease under essential deformation?
- Can we apply this approach to other families of singularities?

## Bibliography


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