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The Transfer Map and Polynomials with Common Roots

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Invariant Theory: Setting and Definitions

Let G a finite group act on $R = \mathbf{k}[x_1, \dots, x_n]$ for some field $\mathbf{k}$.

- The **ring of invariants** is
$$R^G = \{f \in R : g \cdot f = f \text{ for all } g \in G\}.$$
- The **transfer map** is $\mathrm{Tr}^G : R \to R^G, f \mapsto \sum_{g \in G} g \cdot f$.
- The image of Tr^G is an **ideal** of R^G.
$$\mathrm{Im}(\mathrm{Tr}^G) = \begin{cases} R^G & \text{if } |G| \in \mathbf{k}^\times \\ \text{a proper, nonzero ideal} & \text{if } |G| \notin \mathbf{k}^\times \end{cases}$$
- When $G = S_n$ (the symmetric group), the invariant ring R^G is a **polynomial algebra** generated by the **elementary symmetric polynomials** $e_1, \dots, e_n$ of degrees $1, \dots, n$.

$\mathrm{Im}(\mathrm{Tr}^{S_n})$ as an Elimination Ideal

Theorem (Feshbach, 1982)

When $\mathbf{k} = \mathbb{F}_p$ and p divides $|G|$, one has
$$\sqrt{\mathrm{Im}(\mathrm{Tr}^G)} = R^G \cap \bigcap_{\substack{g \in G \\ |g| = p}} \langle x_i - g \cdot x_i : i = 1, \dots, n \rangle.$$

For $G = S_n$, the image of the transfer is a radical ideal. Feshbach's Theorem translates to the following condition on points in $\mathbb{A}^n_{\mathbf{k}}$ (up to permutation):
$$\begin{aligned} (a_1, \dots, a_n) \in V(\mathrm{Im}(\mathrm{Tr}^G)) &\iff \text{at least } p \text{ of the } a_i \text{ are equal} \\ &\iff \text{the polynomial} \\ &\qquad f(t) = (t + a_1)(t + a_2) \cdots (t + a_n) \in \mathbf{k}[t] \\ &\qquad \text{has a root of multiplicity } p. \end{aligned}$$

Corollary (Derksen–P.)

Let $G = S_n$ and $k = \mathbb{F}_p$ with $p \leq n$. Express n uniquely as
$$n = \mathbf{q}p + \mathbf{r}, \qquad 0 \leq \mathbf{q}, \quad 0 \leq \mathbf{r} \leq p - 1.$$
The ideal $\mathrm{Im}(\mathrm{Tr}^{S_n})$ is an elimination ideal
$$\mathrm{Im}(\mathrm{Tr}^{S_n}) = \langle f_0, f_1, \dots, f_{p-1} \rangle \cap R^{S_n},$$
where the $f_i \in R^{S_n}[t]$ satisfy

- *f_0 is **monic** in t of degree $\mathbf{q}$,*
- *$f_1, \dots, f_{\mathbf{r}}$ are **generic** in t of degree $\mathbf{q}$, and*
- *$f_{\mathbf{r}+1}, \dots, f_{p-1}$ are **generic** in t of degree $\mathbf{q} - 1$.*

Example

Let $p = 3$ and $n = 6, 7$. The polynomials f_0, f_1, f_2 from above are:

$(n = 6)$	$(n = 7)$
$f_0 = t^2 + e_3 t + e_6$	$f_0 = t^2 + e_3 t + e_6$
$f_1 = e_1 t + e_4$	$f_1 = e_1 t^2 + e_4 t + e_7$
$f_2 = e_2 t + e_5$	$f_2 = e_2 t + e_5$

Key observation: When $n = 7$, we may replace f_1 with
$$f_1' = f_1 - e_1 f_0 = (e_4 - e_1 e_3)t + (e_7 - e_1 e_6),$$
and the collection of coefficients
$$\{e_4 - e_1 e_3, e_2, e_3, e_7 - e_1 e_6, e_5, e_6\}$$
remains algebraically independent.

Syzygy Stability Theorem

Theorem (Derksen–P.)

For each n, let
$$A_n := \mathbb{F}_p[x_1, \dots, x_n]^{S_n} = \mathbb{F}_p[e_1, \dots, e_n].$$
Fix $q \geq 0$. Then for each $r = 0, 1, \dots, p - 1$, there exists a flat algebra inclusion $\iota_r : A_{qp} \to A_{qp+r}$ such that
$$\mathrm{Im}(\mathrm{Tr}^{S_{qp+r}}) \cong A_{qp+r} \otimes_{A_{qp}} \mathrm{Im}(\mathrm{Tr}^{S_{qp}}) \quad \text{as } A_{qp+r}\text{-modules}.$$
*In particular, the graded minimal free resolution of $\mathrm{Im}(\mathrm{Tr}^{S_{qp+r}})$ **depends only on** q.*

Defining Equations

Conjecture (Derksen–P.)

The ideal $\mathrm{Im}(\mathrm{Tr}^{S_{qp}})$ is generated by the maximal minors of a block-Sylvester matrix
$$X = [X_0 \; X_1 \; \cdots \; X_{p-1}],$$
where X_0 has $q - 1$ columns (with coefficients from f_0) and X_i has q columns (with coefficients from f_i) for each $i = 1, \dots, p - 1$.

Theorem (Derksen–P.)

The conjecture holds for $q = 2$. In particular,
$$\mathrm{Im}(\mathrm{Tr}^{S_{2p}}) = \mathcal{I}_3 \begin{bmatrix} 1 & e_1 & 0 & e_2 & 0 & \cdots & e_{p-1} & 0 \\ e_p & e_{p+1} & e_1 & e_{p+2} & e_2 & \cdots & e_{p+(p-1)} & e_{p-1} \\ e_{2p} & 0 & e_{p+1} & 0 & e_{p+2} & \cdots & 0 & e_{p+(p-1)} \end{bmatrix}.$$

Proof ideas.

- Deduce that $\mathrm{Im}(\mathrm{Tr}^{S_{2p}})$ contains the determinantal ideal J.
- Realize $\mathrm{Im}(\mathrm{Tr}^{S_{2p}})$ as the kernel of the algebra map $A = \mathbf{k}[e_1, \dots, e_{2p}] \xrightarrow{\varphi} \mathbf{k}[u_1, \dots, u_p, \alpha]$ arising from factoring
$$f_0 = t^2 + e_p t + e_{2p} = (t + \alpha)(t + u_p), \quad f_i = e_i t + e_{p+i} = (t + \alpha) u_i$$
and equating coefficients on t.
- Show that $\mathrm{in}_{\prec}(J)$ contains the ideal
$$L := \bigcap_{j=1}^{p+1} \langle e_j, e_{j+1}, \dots, e_{j+p-2} \rangle$$
generated by monomials with no "large gaps" in their support.
- Construct a multigraded bijection
$$\{\text{monomials not in } L\} \leftrightarrow \{\text{monomials in } \mathrm{in}_{\prec'} \varphi(A)\}$$
to conclude the equality of Hilbert series $\mathrm{Hilb}(L, \mathbf{t}) = \mathrm{Hilb}(\mathrm{Im}(\mathrm{Tr}^{S_{2p}}), \mathbf{t})$, and hence $\mathrm{Im}(\mathrm{Tr}^{S_{2p}}) = J$.

$\square$

Betti numbers of $\mathrm{Im}(\mathrm{Tr}^{S_{2p}})$: Summary of Results

We exhibit a minimal, **linear** free resolution of the associated graded ideal of $\mathrm{Im}(\mathrm{Tr}^{S_{2p}})$ (setting $\deg(e_i) = 1$). From this, we compute the projective dimension and total Betti numbers of $\mathrm{Im}(\mathrm{Tr}^{S_{2p}})$.

Bibliography

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