Commutative Algebra and Algebraic Geometry in Milan

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On the quadratic rank of Veronese embeddings in all characteristics

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Rank Index

Any quadratic polynomial f in $S = \mathbb{K}[z_0, \dots, z_N]$ can be written as

$$f \; = \; z_0^2 + z_1^2 + \cdots + z_r^2 \quad (r \le N)$$

after a suitable change of basis, provided that $\operatorname{char}(\mathbb{K}) \neq 2$. The **quadratic rank** of f is defined as the number of these squares.

Let X be a projective scheme and $\mathcal{L}$ a very ample line bundle on X inducing the linearly normal embedding

$$X \;\subset\; \mathbb{P}H^0(X, \mathcal{L}) = \mathbb{P}^N.$$

We say that $\mathcal{L}$ satisfies **property QR(k)** if its defining ideal $I(X, \mathcal{L})$ can be generated by quadratic forms of rank at most k. In particular, if $\mathcal{L}$ is determinantally presented, then it satisfies property QR(4).

Definition

The **rank-index** of $(X, \mathcal{L})$ is the minimum integer k such that $\mathcal{L}$ satisfies property QR(k).

Our goal is to examine the rank-index of Veronese varieties from various perspectives in all characteristics.

Main Theorem I

Theorem (Han-Lee-Moon-Park [1])

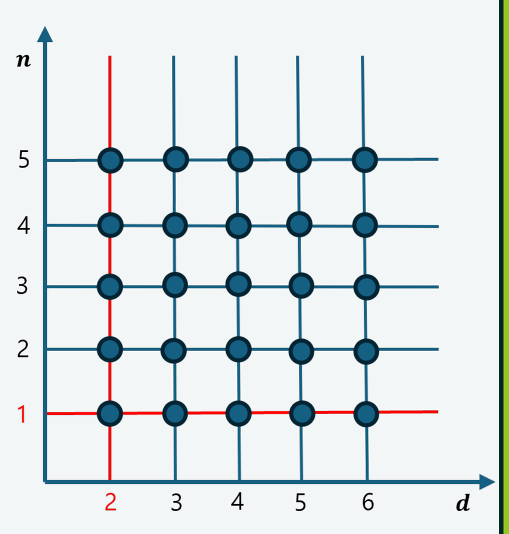
When $\operatorname{char}(\mathbb{K}) \neq 2, 3$,

rank-index $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = 3$ *if* $n \ge 1, d \ge 2$.

When $\operatorname{char}(\mathbb{K}) = 3$,

rank-index $(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = 3$ *if* $d \ge 2$.

rank-index $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2)) = 3$ *iff* $n \le 2$.



$Char(\mathbb{K}) \neq 2, 3$

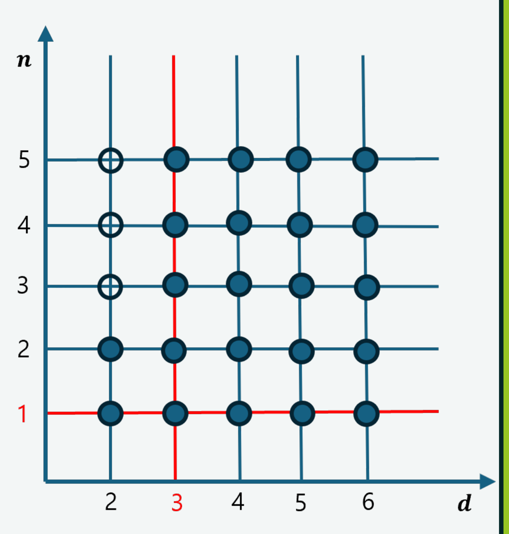
Theorem (Lee-Park-Sim [2])

When $\operatorname{char}(\mathbb{K}) = 3$,

rank-index $(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) = 3$ *if* $n \ge 2, d \ge 3$.

Furthermore, if $X = \nu_2(\mathbb{P}^n)$ *is the second Veronese variety of* $\mathbb{P}^n$*, then*

$\operatorname{codim}(\langle \Phi_3(X) \rangle, \mathbb{P}(I(X)_2)) = \binom{n+1}{4}$.



$Char(\mathbb{K}) = 3$

Rank 3 Construction

By employing the mapping introduced in [1], property QR(3) can be systematically analyzed. In particular, let $X = \nu_d(\mathbb{P}^n)$ be the d-th Veronese variety. For each integer $1 \le \ell \le \frac{d}{2}$, we define the map

$$Q_\ell : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\ell)) \times H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\ell)) \times H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d - 2\ell)) \to I(X)_2$$

by

$$Q_\ell(f, g, h) = \varphi(f \otimes f \otimes h)\varphi(g \otimes g \otimes h) - \varphi(f \otimes g \otimes h)^2,$$

where φ is the natural $\mathbb{K}$-isomorphism between $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ and $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$.

Key Strategy: Since every element in the image of Q_ℓ is a quadratic form of rank at most 3 in $I(X)$, it is enough to show that all generators of $I(X)$ are contained in the spanning set of $\operatorname{Im}(Q_\ell)$ to establish property QR(3).

Rank Locus

Let $\Phi_k(X)$ be the set of all quadratic forms of rank at most k in the ideal $I(X)$. Then $\Phi_k(X)$ is indeed an algebraic set with a natural nested structure:

$$\Phi_1(X) \;\subset\; \Phi_2(X) \;\subset\; \cdots \subset\; \Phi_k(X) \;\subset\; \Phi_{k+1}(X) \;\subset\; \cdots.$$

Investigating the geometric structure of the rank-k locus $\Phi_k(X)$ provides a direct approach to determining property QR(k).

Main Theorem II

Theorem (Park [3])

For each integer $1 \le \ell \le \frac{d}{2}$*, the map* Q_ℓ *induces a finite morphism*

$$\widetilde{Q_\ell} : \mathbb{G}(1, \mathbb{P}^p) \times \mathbb{P}^q \to \mathbb{P}\left(I(\nu_d(\mathbb{P}^n))_2\right)$$

where $p = h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\ell)) - 1$ *and* $q = h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d - 2\ell)) - 1$.

Denoting its image by $W_\ell(\nu_d(\mathbb{P}^n))$*, we have*

$$\Phi_3(\nu_d(\mathbb{P}^n)) \;=\; \bigcup_{1 \le \ell \le \frac{d}{2}} W_\ell(\nu_d(\mathbb{P}^n)),$$

which gives an irreducible decomposition of $\Phi_3(\nu_d(\mathbb{P}^n))$*, and the dimension of each component* W_ℓ *is* $2p + q - 2$.

Theorem (Park-Sim [4])

1. *For* $n = 1$*, all components* W_ℓ *are nondegenerate. In particular, this implies property QR(3).*
2. *For* $n \ge 1$*, the variety* $\mathbb{G}(1, \mathbb{P}^p) \times \mathbb{P}^q$ *is the normalization and a de-singularization of* W_ℓ*.*

Example

We can compute the image under $\widetilde{Q_\ell}$ of a point $(\langle f, g \rangle, h)$ in $\mathbb{G}(1, \mathbb{P}^p) \times \mathbb{P}^q$, where f, g and h are general forms of degree ℓ and $d - 2\ell$ respectively. For instance, let $n = 3$ and $d = 2$, which is the smallest case where property QR(3) fails in $\operatorname{char}(\mathbb{K}) = 3$.

Basis	Coefficient
$Z_{(2,0,0,0)}Z_{(0,2,0,0)} - Z_{(1,1,0,0)}Z_{(1,1,0,0)}$	p_{01}^2
$Z_{(2,0,0,0)}Z_{(0,1,1,0)} - Z_{(1,1,0,0)}Z_{(1,0,1,0)}$	$2p_{01}p_{02}$
$Z_{(1,1,0,0)}Z_{(0,0,1,1)} - Z_{(1,0,0,1)}Z_{(0,1,1,0)}$	$2p_{02}p_{13} \;+\; 2p_{03}p_{12}$
$Z_{(1,0,1,0)}Z_{(0,1,0,1)} - Z_{(1,0,0,1)}Z_{(0,1,1,0)}$	$2p_{02}p_{13} \;-\; 4p_{03}p_{12}$

Note: p_{ij} is the Plücker coordinate induced by $\mathbb{G}(1, \mathbb{P}^p)$.

When the index sum is $(1, 1, 1, 1)$, the above coefficients become linearly dependent exclusively in characteristic 3. Consequently, $\Phi_3(\nu_2(\mathbb{P}^3))$ becomes degenerate.

Bibliography

1. K. Han, W. Lee, H. Moon, and E. Park, Rank 3 quadratic generators of Veronese embeddings, Compos. Math. 157 (2021), 2001–2025.
2. D. Lee, E. Park, and S. Sim, Rank 3 quadratic generators of Veronese embeddings: the characteristic 3 case, to appear in J. Algebra.
3. E. Park, On rank 3 quadratic equations of projective varieties, Trans. Amer. Math. Soc. 377(3) (2024), 2049–2064.
4. E. Park, and S. Sim, On rank 3 quadratic equations of Veronese embeddings, Math. Nachr. 298 (2025), 3135-3155.

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