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The Homology of the Hilbert Scheme of Points

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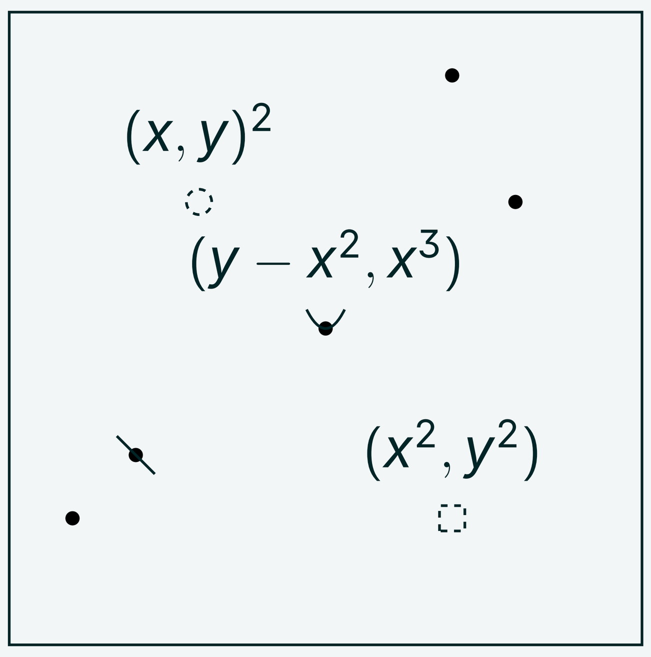
Definitions/conventions

We work over the complex numbers $\mathbb{C}$ and in the analytic topology, when discussing topology.

- The **Hilbert scheme of d points in $\mathbb{A}^n$** is the scheme
$$\mathrm{Hilb}_d(\mathbb{A}^n) = \{X \subseteq \mathbb{A}^n_{\mathbb{C}} \text{ closed} \mid \dim(X) = 0, \dim_{\mathbb{C}}(\mathcal{O}(X)) = d\}.$$
Its points can also be viewed as ideals $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$.
- The **punctual** Hilbert scheme of d points in $\mathbb{A}^n$ is
$$\mathrm{Hilb}_d(\mathbb{A}^n, 0) = \{X \in \mathrm{Hilb}_d(\mathbb{A}^n) \mid X \subseteq \{0\}\}.$$
- For a fixed $h : \mathbb{N} \longrightarrow \mathbb{N}$, the **Hilbert–Samuel stratum** is
$$H^n_h = \{X \in \mathrm{Hilb}_d(\mathbb{A}^n, 0) \mid \dim_{\mathbb{C}} \mathfrak{m}^i/\mathfrak{m}^{i+1} = h(i)\},$$
so $\mathrm{Hilb}_d(\mathbb{A}^n, 0) = \coprod_h H^n_h$. For instance, $h(1) = \dim T_0 X$.
- $\mathcal{H}^n_h = \{X \in H^n_h \mid I(X) \text{ homogeneous}\}$.
- The **initial ideal** of an ideal I, denoted $\mathrm{in}(I)$, is the ideal generated by the lowest degree terms of the elements of I.

An element



$\in \mathrm{Hilb}_{15}(\mathbb{A}^2)$

Main theorem

Theorem (S., [2])

$H_i(\mathrm{Hilb}_5(\mathbb{A}^n); \mathbb{Z})$ is free, 0 for i odd, and has Poincaré polynomial:
$$\begin{aligned} p_{\mathrm{Hilb}_5(\mathbb{A}^n)} = {} & z^{2(3n-3)} g_{1,n}(z) \\ & + z^{4(n-1)}(1+z^2) g_{2,n}(z) \\ & + z^{2(2n-4)} g_{1,3}(z) g_{2,n}(z) \\ & + z^{2(n-3)} g_{1,6}(z) g_{3,n}(z) \\ & + g_{4,n}(z), \end{aligned}$$
where $g_{k,n}(z)$ is the Poincaré polynomial of $\mathrm{Gr}_k(\mathbb{C}^n)$.

Reduction

Theorem (Totaro, [3])

The inclusion $\mathrm{Hilb}_d(\mathbb{A}^n, 0) \longrightarrow \mathrm{Hilb}_d(\mathbb{A}^n)$ is a homotopy equivalence.

Fibrations

$$\begin{tikzcd} \mathbb{A}^{(n-h(1))(d-h(1)-1)} \arrow[r] & F \arrow[r] \arrow[d, "\pi"] & H^n_h \arrow[d, "\tau"] \\ & H^{h(1)}_h \arrow[d, "\mathrm{in}"] & \mathrm{Gr}_{n-h(1)}(\mathbb{C}\langle x_1, \ldots, x_n\rangle) \\ & \mathcal{H}^{h(1)}_h & \end{tikzcd}$$

Fibration 1. $\tau : H^n_h \longrightarrow \mathrm{Gr}_{n-h(1)}(\mathbb{C}\langle x_1, \ldots, x_n\rangle)$

- **Definition.** $\tau(I) = \mathrm{in}(I)_1$.
- A basis for $\mathrm{in}(I)_1$ defines linear equations for $T_0(X) \subseteq T_0(\mathbb{A}^n)$.
- τ is Zariski locally trivial, due to [1].
- Take coordinates $x_1 \ldots x_{h(1)}, y_1, \ldots, y_{n-h(1)}$ and take the fiber F over $\mathrm{span}(\{y_i\})$.

Fibration 2. $\pi : F \longrightarrow H^{h(1)}_h$

- **Definition.** $\pi(I) = I \cap \mathbb{C}[x_1, \ldots, x_{h(1)}]$ – the *elimination ideal*. That is, $X \mapsto$ the projection of X onto $T_0(X)$.
- π is a vector bundle, due to [2].
- We view $\pi(I)$ as an ideal of $\mathbb{C}[x_1, \ldots, y_1, \ldots]$ by adding $(y_1, \ldots, y_{n-h(1)})$. That is, embedding $T_0(X) \subseteq \mathbb{A}^n$.
- The affine space fibers come from perturbing the y_i by higher order monomials with arbitrary coefficients.

"Fibration" 3. $\mathrm{in} : H^{h(1)}_h \longrightarrow \mathcal{H}^{h(1)}_h$

- **Definition.** $I \mapsto \mathrm{in}(I)$ defines a morphism $\mathrm{in} : H^{h(1)}_h \longrightarrow \mathcal{H}^{h(1)}_h$.
- Let t be the maximal so that $h(t) \neq 0$. in is a vector bundle **if** $\mathrm{in}(I)_j = 0$ for $j \leq t-2$ (a proof is recorded in [2]).
- The affine space fibers come from perturbing elements of $\mathrm{in}(I)_{t-1}$ by degree t monomials with arbitrary coefficients.

How to build an ideal

We will construct an ideal in $H^3_{(1,2,1,1)}$ with tangent space $\{y_1 = 0\}$.

- There is an isomorphism $\mathbb{P}(\mathbb{C}\langle \partial_{x_1}, \partial_{x_2}\rangle) \longrightarrow \mathcal{H}^2_{(1,2,1,1)}$ sending $[\alpha] \mapsto (x_1, x_2)^4 + (f \mid \alpha^2(f) = 0)$. For $\alpha = \partial_{x_1}$, we get
$$(x_1, x_2)^4 + (x_1 x_2, x_2^2) \in \mathcal{H}^2_{(1,2,1,1)}.$$
- in is a vector bundle, and we perturb x_1^2 and $x_1 x_2$ by cubics.
$$(x_1, x_2)^4 + (x_1 x_2 + \mathbf{3x_1^3} + \mathbf{4x_1^2 x_2}, x_2^2 - \mathbf{x_1 x_2^2}) \in H^2_{(1,2,1,1)}.$$
- We view this as an ideal in $\mathbb{C}[x_1, x_2, y_1]$ via
$$(\mathbf{y_1}) + (x_1, x_2)^4 + (x_1 x_2 + 3x_1^3 + 4x_1^2 x_2, x_2^2 - x_1 x_2^2) \in H^2_{(1,2,1,1)}.$$
- π is a vector bundle, and we perturb y_1 by higher order terms.
$$(y_1 - \mathbf{x_1^2}) + (x_1, x_2)^4 + (x_1 x_2 + 3x_1^3 + 4x_1^2 x_2, x_2^2 - x_1 x_2^2) \in H^3_{(1,2,1,1)}.$$

All together now

Each summand in $p_{\mathrm{Hilb}_5(\mathbb{A}^n)}$ is the Poincaré polynomial of a Hilbert–Samuel stratum, which can be computed via fibrations.

Bibliography

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[2] Jas Singh. "Extreme Calabi–Yau Varieties and the Topology of the Hilbert Scheme of Points". PhD thesis. UCLA, 2026.

[3] Burt Totaro. "Torus actions, Morse homology, and the Hilbert scheme of points on affine space". In *Épijournal de Géométrie Algébrique Volume 5* (Aug. 2021).

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