Chitra Venugopal

Free Resolutions of Symmetric Algebras of Ideals with Deviation Two

Chennai Mathematical Institute chitrav@cmi.ac.in

Abstract

For a graded ideal I in a graded ring, the deviation of I is defined as the difference between the minimal number of generators of I and its grade. In this work, we construct bigraded free resolutions of the symmetric algebras of certain classes of ideals of deviation two and provide some combinatorial motivation for defining these classes. This is joint work with Neeraj Kumar and Aniruddha Saha.

Construction

- Let I be a deviation two ideal in a polynomial ring $B = K[X]$ over a field K, minimally generated by n elements. Let Z_i denote the i-th Koszul cycle of I and set $S = B[Y_1, \dots, Y_n]$.
- Cid-Ruiz, Polini, and Ulrich [2] constructed a complex $\mathcal{J}_\bullet$ which is acyclic whenever the approximation complex $\mathcal{Z}_\bullet$ is acyclic. Moreover, $H_0(\mathcal{J}_\bullet) \cong H_0(\mathcal{Z}_\bullet) \cong \mathrm{Sym}_B(I)$.

$$\mathcal{Z}_\bullet : \quad 0 \to Z_{n-1} \otimes_B S(-(n-1)) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} Z_1 \otimes_B S(-1) \xrightarrow{\partial_1} Z_0 \otimes_B S \to 0$$

$$\mathcal{J}_\bullet : \quad 0 \to \xrightarrow{\sigma_4} C_{n-4} \to \cdots \to C_1 \xrightarrow{\sigma_1} C_0 \xrightarrow{\eta} Z_2 \otimes_B S(-2) \xrightarrow{\partial_2} Z_1 \otimes_B S(-1) \xrightarrow{\partial_1} S \to 0$$

  where each C_i is a free S-module.
- From the short exact sequence
$$0 \longrightarrow B_2 \longrightarrow Z_2 \longrightarrow H_2 \longrightarrow 0,$$
  the study of the module Z_2 reduces to that of the Koszul boundary module B_2 and the Koszul homology module H_2.

Motivation and Main Question

Setup. Let I_0 be a deviation two ideal in a polynomial ring B. For $k \geq 1$, define
$$I_k = I_0 + \langle g_1, \dots, g_k \rangle,$$
where $\{g_1, \dots, g_k\}$ is a B/I_0-regular sequence. Then each I_k is again a deviation two ideal.

Question

When is the sequence $\{g_1, \dots, g_k\}$ regular on the first Koszul homology module $H_1(K_\bullet(I_0))$?

Why is this important? If the above question has a positive answer, then for every $k \geq 1$, the second Koszul homology module of I_k can be described in terms of $H_2(K_\bullet(I_0))$.

Classes of Ideals Satisfying the Property

Let $I_0 = \langle f_1, \dots, f_n \rangle$ be a deviation two ideal in a polynomial ring B, where $\{f_1, \dots, f_{n-2}\}$ is a regular sequence. Set $I'_0 = \langle f_1, \dots, f_{n-2} \rangle$, $I''_0 = \langle f_1, \dots, f_{n-1} \rangle$, and assume $(I'_0 : f_{n-1}) \subseteq (I'_0 : f_n)$.
For $k \geq 1$, define
$$I_k = I_0 + \langle g_1, \dots, g_k \rangle,$$
where $\{g_1, \dots, g_k\}$ is both B/I_0-regular and B/I''_0-regular.

Class I. Assume further that
$$(I'_0 + \langle g_1, \dots, g_k \rangle : f_{n-1}) \subseteq (I'_0 + \langle g_1, \dots, g_k \rangle : f_n).$$

Class II. Assume further that $\{g_1, \dots, g_k\}$ is regular on $M = (I''_0 : f_n)$.

Both these families of ideals answer the above question in positive.

Main Result: Bigraded Free Resolution of $\mathrm{Sym}_B(I_k)$

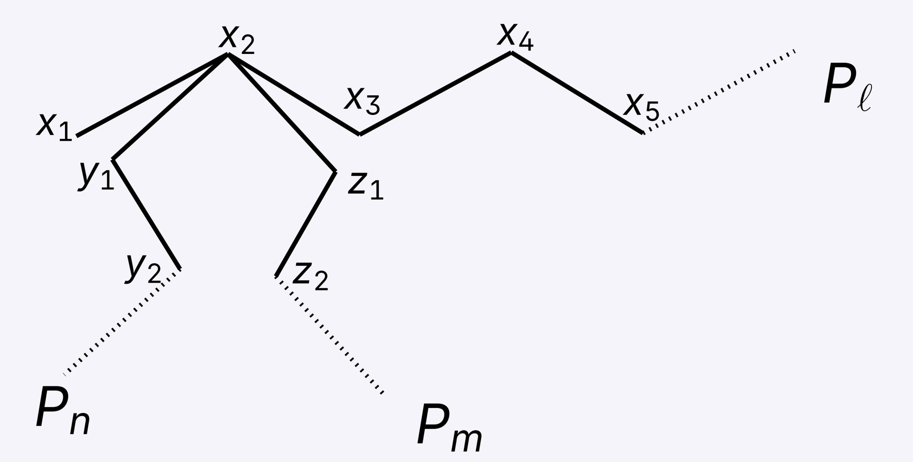
Notation. Let $\deg(g_i) = d_i$. Further, let
$$(\mathbb{F}^{(k)}_\bullet, \varphi^{(k)}_\bullet), \qquad (K^{(k)}_\bullet, \nu^{(k)}_\bullet), \qquad (\mathfrak{L}^{(k)}_\bullet, \tau^{(k)}_\bullet)$$
denote graded free resolutions of B/I_k, the Koszul complex $K_\bullet(I_k)$, and $H_2(K_\bullet(I_k))$, respectively.

Theorem. Let I_k be an ideal arising from the construction in **Setup**, and assume that $\{g_1, \dots, g_k\}$ is $H_1(K_\bullet(I_0))$-regular. Let $S = B[Y]$ be a bigraded polynomial ring, and assume that the approximation complex $\mathcal{Z}_\bullet(I_k, B)$ is acyclic for each $k \geq 1$. Then $\mathrm{Sym}_B(I_k)$ admits a bigraded free resolution
$$\cdots \xrightarrow{\delta_3} \mathfrak{D}_2 \xrightarrow{\delta_2} \mathfrak{D}_1 \xrightarrow{\delta_1} \mathfrak{D}_0 \longrightarrow 0$$
Here $\mathfrak{D}_0 = S$, and for $i \geq 1$,
$$\mathfrak{D}_i = C^{(k)}_{i-3} \oplus \left(K^{(k)}_{i+1} \oplus \mathfrak{L}^{(k)}_{i-2}\right) \otimes S(-2) \oplus \mathbb{F}^{(k)}_{i+1} \otimes S(-1).$$
Moreover,
$$C^{(k)}_{i-3} = \bigoplus_{r=0}^{i-3} \left(\bigoplus_{1 \leq j_1 < \cdots < j_{r+3} \leq \mu(I_k)} S\left(-\sum_{\ell=1}^{r+3} d_{j_\ell}, -r-3 \right) \right).$$
The differentials $\{\delta_i\}_{i \geq 1}$ are obtained via an iterative mapping-cone construction.

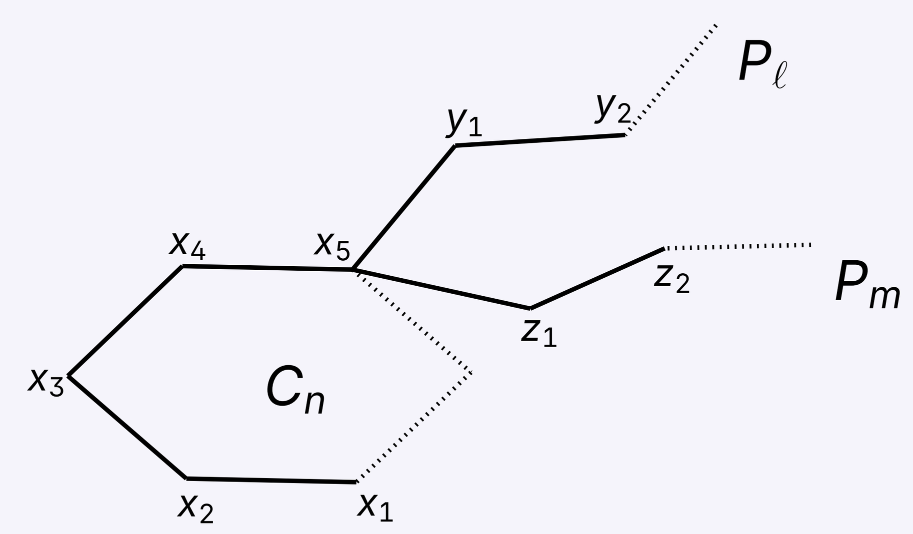
Applications

- For $r \geq 1$, let P_r denote the path graph on r vertices and C_r denote the cycle graph on r vertices.

  $G_{1,(\ell,m,n)}$:



  $G_{2,(\ell,m,n)}$:



- The corresponding binomial edge ideals belong to **Class I**. Consequently, the previous theorem yields explicit bigraded free resolutions of their symmetric algebras.

References

[1] N. Kumar, A. Saha, and C. Venugopal: Free Resolutions of Symmetric Algebras of Ideals with Deviation Two. (arXiv:2605.27632), 2026.
[2] Y. Cid-Ruiz, C. Polini, and B. Ulrich: Generalized jouanolou duality, weakly gorenstein rings, and applications to blowup algebras. J. Reine Angew. Math, 826:225–270, 2025.

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