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Maximal Cohen–Macaulay Modules on Symplectic Singularities

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Motivation: Reflexive modules on ADE singularities

By the work of Artin–Verdier, for an ADE surface singularity and its minimal resolution, there is a one-to-one correspondence

$$\left\{\begin{matrix}\text{indecomposable non-free}\\ \text{reflexive modules}\end{matrix}\right\} \leftrightarrow \left\{\begin{matrix}\text{irreducible components of}\\ \text{exceptional fiber}\end{matrix}\right\}.$$

ADE singularities are the basic two-dimensional symplectic singularities, where reflexive and maximal Cohen–Macaulay (abbreviated as MCM) modules coincide. They motivate the study of MCM sheaves on higher dimensional symplectic singularities, especially those admitting a symplectic resolution.

Approach using Grothendieck duality

For a symplectic resolution $\pi : \widetilde{X} \to X$ of an affine singular variety, one has $\pi_*\mathcal{O}_{\widetilde{X}} = \mathcal{O}_X$ and $\pi^!\mathcal{O}_X = \mathcal{O}_{\widetilde{X}}$. If $\mathcal{F}_0$ is a reflexive sheaf on X and $\mathcal{F} := (\pi^*\mathcal{F}_0)^{\vee\vee}$, then $\pi_*\mathcal{F} = \mathcal{F}_0$. Hence the problem is to determine when a reflexive sheaf on $\widetilde{X}$ pushes forward to a MCM sheaf on X. Grothendieck duality gives the spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i_{\mathcal{O}_X}(R^{-j}\pi_*\mathcal{F}, \mathcal{O}_X) \Longrightarrow \mathrm{Ext}^{i+j}_{\mathcal{O}_{\widetilde{X}}}(\mathcal{F}, \mathcal{O}_{\widetilde{X}}).$$

In particular, the vanishings $R^{>0}\pi_*\mathcal{F} = R^{>0}\pi_*\mathcal{F}^\vee = 0$ imply that $\mathcal{F}_0 = \pi_*\mathcal{F}$ is MCM. Conversely, for vector bundles on the isolated symplectic singularities, this criterion is almost necessary. Indeed, suppose $\mathcal{F}$ is a vector bundle on $\widetilde{X}$, composing local duality $(R^n\pi_*\mathcal{F})' \cong \mathrm{Ext}^{2n}_{\mathcal{O}_X}(R^n\pi_*\mathcal{F}, \mathcal{O}_X)$ with the edge map

$$\mathrm{Ext}^{2n}_{\mathcal{O}}(R^n\pi_*\mathcal{F}, \mathcal{O}) = E_2^{2n,-n} \to E_\infty^{2n,-n} \hookrightarrow \mathrm{Ext}^n_{\widetilde{\mathcal{O}}}(\mathcal{F}, \widetilde{\mathcal{O}}) = H^n(\mathcal{F}^\vee)$$

we obtain a natural map $e_{\mathcal{F}} : (R^n\pi_*\mathcal{F})' \longrightarrow R^n\pi_*\mathcal{F}^\vee$

Theorem. Let X be a $2n$-dimensional variety with isolated singularities, and let $\pi : \widetilde{X} \to X$ be a symplectic resolution. For a vector bundle $\mathcal{F}$ on $\widetilde{X}$, $\pi_*\mathcal{F}$ is maximal Cohen–Macaulay if and only if

1. $R^i\pi_*\mathcal{F} = R^i\pi_*\mathcal{F}^\vee = 0$ for $1 \le i \le n-1$;
2. $e_{\mathcal{F}} : (R^n\pi_*\mathcal{F})' \to R^n\pi_*\mathcal{F}^\vee$ is an isomorphism.

Application to the resolution $T^*\mathbb{P}^n \to \mathcal{N}_{n+1,1}$

Denote by $\mathcal{N}_{n+1,1}$ the variety of nilpotent $(n+1)\times(n+1)$ matrices of rank at most 1. It admits a Springer-type symplectic resolution $\pi : T^*\mathbb{P}^n \longrightarrow \mathcal{N}_{n+1,1}$. Let D be a generator of $\mathrm{Cl}(\mathcal{N}_{n+1,1}) \cong \mathbb{Z}$. Applying the theorem above to Weil divisors on $\mathcal{N}_{n+1,1}$, we find that $\mathcal{O}(kD)$ is MCM if and only if $-n \le k \le n$.

To construct indecomposable MCM sheaves with higher ranks, we pull back indecomposable vector bundles E along $p : T^*\mathbb{P}^n \to \mathbb{P}^n$ and impose $R^{>0}\pi_*p^*E = R^{>0}\pi_*p^*E^\vee = 0$. This is equivalent to $H^{>0}(E(m)) = H^{<n}(E(-m-n-1)) = 0$ for $m \ge 0$. For $n = 2$ and $\mathrm{rank}(E) = 2$, this holds exactly in three cases:

1. $E = T_{\mathbb{P}^2}(-1)$ or $T_{\mathbb{P}^2}(-2)$.
2. E is the unique nontrivial extension of I_x by $\mathcal{O}_{\mathbb{P}^2}$ for some $x \in \mathbb{P}^2$.
3. E is a stable 2-bundle with $c_1(E) = 0$, $c_2(E) = 2$.

One can find stable (hence indecomposable) bundles with correct cohomology vanishing in the family of Steiner bundles and their twists. Steiner bundles are those defined by short exact sequence $0 \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus t} \to \mathcal{O}_{\mathbb{P}^n}^{\oplus t+r} \to E \to 0$. In this way, one can show that:

Theorem. For $r \ge n$, there is an integer M_r such that for $k \ge M_r$, there exists an indecomposable MCM module on $\mathcal{N}_{n+1,1}$ of rank kr. If $n|r$ or $n = 2$, we can set $M_r = 1$.

Equivariant MCM sheaves on minimal orbit closures

The huge family of MCM sheaves on $\mathcal{N}_{n+1,1}$ illustrates the difficulty of classifying such objects. However, after restricting to $SL_{n+1}(\mathbb{C})$-equivariant sheaves whose restrictions to the smooth locus are semisimple, a classification is possible, and we generalize this to all minimal nonzero nilpotent orbit closures of simple Lie algebras.

Let G be a complex simply connected Lie group with simple Lie algebra $\mathfrak{g}$. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, a root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha\in\Phi}\mathfrak{g}_\alpha$, and a choice of simple roots $\Delta \subset \Phi$. Let θ be the highest root. Choose root vectors $e_\theta \in \mathfrak{g}_\theta$ and $f_\theta \in \mathfrak{g}_{-\theta}$, and let $h_\theta \in \mathfrak{h}$ be the corresponding coroot such that $\mathbb{C}\langle e_\theta, f_\theta, h_\theta\rangle \cong \mathfrak{sl}_2$. The adjoint action of h_θ gives a five-step grading

$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

Moreover, $\mathfrak{g}_0 = \mathfrak{g}^\natural \oplus \mathbb{C}h_\theta$, where $\mathfrak{g}^\natural$ is the centralizer of $\mathbb{C}\langle e_\theta, f_\theta, h_\theta\rangle$. Suppose H is the stabilizer of e_θ under the adjoint action and P is the stabilizer of $\mathbb{C}e_\theta$. Denote the Levi factor of H by L, then there is

$$\mathrm{Lie}(H) = \mathfrak{g}^\natural \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2, \quad \mathrm{Lie}(P) = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2, \quad \mathrm{Lie}(L) = \mathfrak{g}^\natural$$

One sees that P is parabolic and hence $X = G/H \cup \{0\}$ is the minimal orbit closure. An equivariant MCM sheaf restricts and is the extension of an equivariant bundle $\mathcal{E}_V = G\times_H V^\vee$ on G/H. By computing local cohomology sequence, we have

$$\text{extension } \overline{\mathcal{E}_V} \text{ is MCM} \Longleftrightarrow H^i(\mathcal{E}_V, G/H) = 0,\ 1 \le i \le \dim G/H - 2$$

An irreducible representation V_λ of H is determined by a dominant integral weight λ of its Levi factor L. Suppose $S = \exp(\mathbb{C}h_\theta)$, then S normalize H, and $S \cong \mathbb{G}_m$, $\quad S\cap H = \mu_2$, $\quad SH = P$. Hence $q : G/H \to G/P$ is an affine map with fibers S/μ_2. Then

$$q_*\mathcal{E}_{V_\lambda} \cong \bigoplus_{2|(j-\mathrm{wt}_{\mu_2})} G\times_H (V_\lambda^\vee \otimes \mathbb{C}t^{-j}) \cong \bigoplus_{2|(j-\mathrm{wt}_{\mu_2})} \mathcal{E}_{\lambda+\frac{j}{2}\theta}$$

Since P is parabolic, we can compute the cohomology of $\mathcal{E}_V$ by pushing forward and applying BWB theroem.

Application to $\mathcal{N}_{n+1,1}$

One can apply this story to $\mathcal{N}_{n+1,1}$ and find out that the conclusion on Weil divisors coincides with our previous results.

Theorem. Suppose $\mathcal{O}_{\min}$ is the minimal orbit in $\mathfrak{sl}_{n+1}$ $(n \ge 2)$ with respect to the conjugation action of $SL(n+1)$ and H is the stabilizer. Given an equivariant bundle $\mathcal{E}_V = SL(n+1)\times_H V^\vee$ corresponding to an irreducible representation V of H determined by a weight $(\lambda, \lambda_1, \lambda_2 \ldots \lambda_{n-1})$ in $\mathbb{Z}^n/\mathbb{Z}(0,1,\ldots,1)$ with $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_{n-1}$, the pushforward $i_*\mathcal{E}_V$ is maximal Cohen-Macaulay on orbit closure $\overline{\mathcal{O}_{\min}} = \mathcal{N}_{n+1,1}$ if and only if

1. If $\lambda_1 = \lambda_2 = \cdots = \lambda_{n-1} = \mu$, then $2\mu - n \le \lambda \le 2\mu + n$.
2. If $\lambda_1 > \lambda_{n-1}$, there is $\lambda_{n-1} + \lambda_M - M \le \lambda \le \lambda_1 + \lambda_m + n - m$ and
$$\lambda \in \bigcap_{\alpha\in A}\{\lambda_i + \alpha - i \mid 1 \le i \le n-1\}$$

where $M = \max\{i \mid 1 \le i \le n-1, \lambda_i = \lambda_1\}$, $m = \min\{i \mid 1 \le i \le n-1, \lambda_i = \lambda_{n-1}\}$ and $A = \{N \mid N \in \mathbb{Z}, \lambda_{n-1} + 1 < N < \lambda_1 + n - 1, N \ne \lambda_i + n - i$ for any $1 \le i \le n-1, N \ne \frac{\lambda+n}{2}\}$

Bibliography

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