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Syzygies of virtually linear ideals

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Introduction

Let $S=\mathbb{C}[x_0,x_1,\dots,x_n]$ be the polynomial ring, $\mathfrak{m}$ the maximal ideal generated by the variables, and I a homogeneous ideal generated in degree d. We say that the resolution of I is **linear for** p **steps** if its first linear strand

$$\cdots \to S(-d-p-1)^{b_{p+1,d}} \to S(-d-p)^{b_{p,d}} \to \cdots \to S(-d)^{b_{0,d}} \to I \to 0$$

is exact up to cohomological degree $-p+1$. Equivalently, for every $0\le i\le p$, we have

$$\beta_{i,j}(I)=\dim_{\mathbb{C}}\left(\operatorname{Tor}_i^S(I,\mathbb{C})_{i+j}\right)\neq 0 \iff j=d.$$

This notion measures the complexity of the ideal I. As p increases, the minimal free resolution of I should become simpler. We assume throughout that I is $\mathfrak{m}$-primary and generated in degree d.

Powers of I

Eisenbud, Huneke, and Ulrich conjectured the following:
Conjecture (Eisenbud-Huneke-Ulrich [2]): If the resolution of I is linear for p steps, then

$$I^t=\mathfrak{m}^{td} \quad \text{for all } t\ge \frac{n}{p}.$$

The conjecture is known to hold when I is a monomial ideal [2, Theorem 8.1], when $n=2$ [2, Corollary 7.7], and when $p\ge n/2$ [2, Corollary 7.7]. However, little is known beyond these cases.

Regularity of I

On the other hand, it is interesting to study the Castelnuovo–Mumford regularity of ideals with partially linear resolutions. Assume that the resolution of I is linear for p steps. In the case $p=0$, Macaulay proved that

$$\operatorname{reg}(S/I)\le (n+1)(d-1).$$

At the other extreme, when $p=n$, I has a linear resolution, and hence

$$\operatorname{reg}(S/I)=d-1.$$

More generally, when $p\ge (n-1)/2$, that is, when at least half of the minimal free resolution is linear, Eisenbud, Huneke, and Ulrich proved in [2, Corollary 3.2] that

$$\operatorname{reg}(S/I)\le 2(d-1).$$

Virtually linear ideals

We say that the resolution of I is **virtually linear for** p **steps** if the sheafification of its first linear strand is exact up to cohomological degree $-p+1$, see also [1]. If the resolution of I is linear for p steps, then it is also virtually linear for p steps because sheafification preserves exactness.

Theorem 1 (Y.)

If the resolution of I is virtually linear for p steps, then

$$I^t=\mathfrak{m}^{td} \quad \text{for all } t\ge \left\lfloor \frac{n}{p+1}\right\rfloor + \left\lceil \frac{n-1}{p}\right\rceil.$$

Theorem 2 (Y.)

If the resolution of I is virtually linear for p steps, then

$$\operatorname{reg}(S/I)\le \left\lceil \frac{n+1}{p+1}\right\rceil (d-1).$$

Moreover, this bound is sharp for $p=0$, $p=1$, and $p\ge (n-1)/2$.

Theorem 3 (Y.)

If the resolution of I is virtually linear for 1 step, then

$$\mu(I)\ge nd+1.$$

Moreover, the bound is sharp.

Example

One important class of partially virtually linear ideals naturally arises in the study of binary forms, see [3]. Let $S=\operatorname{Sym}(\operatorname{Sym}^n(\mathbb{C}^2))$ where $\operatorname{Sym}^n(\mathbb{C}^2)$ is identified with the space of binary forms of degree n, and let $I_{d,n}$ be the ideal generated by $\operatorname{Sym}^{nd}(\mathbb{C}^2)\subseteq \operatorname{Sym}^d(\operatorname{Sym}^n(\mathbb{C}^2))=S_d$. It is a fact that the resolution of $I_{d,n}$ is virtually linear for 1 step, but $I_{d,n}$ is not linearly presented in general. Theorem 1 and Theorem 2 imply that

$$I_{d,n}^t=\mathfrak{m}^{td} \quad \text{for all } t\ge \frac{3n}{2}-1 \quad \text{and} \quad \operatorname{reg}(S/I_{d,n})\le \left\lceil\frac{n+1}{2}\right\rceil (d-1).$$

In fact, for this specific class, [3] proves the stronger results

$$I_{d,n}^t=\mathfrak{m}^{td} \quad \text{for all } t\ge n \quad \text{and} \quad \operatorname{reg}(S/I_{d,n})=\left\lceil\frac{n+1}{2}\right\rceil (d-1).$$

Notice that this class gives the optimal examples for Theorem 2 and Theorem 3 when $p=1$.

Future directions

- How can we push Theorem 1 closer to the Eisenbud–Huneke–Ulrich conjecture? Should we expect the Eisenbud–Huneke–Ulrich conjecture for virtually linear ideals?
- We expect Theorem 2 to be sharp in general. How can we find optimal examples for other values of p?
- Prove sharp lower bound on the number of generators for virtually linear ideals in general.

Bibliography

1. Christine Berkesch, Daniel Erman, and Gregory G. Smith, *Virtual resolutions for a product of projective spaces*, Algebr. Geom. **7** (2020).
2. David Eisenbud, Craig Huneke, and Bernd Ulrich, *The regularity of Tor and graded Betti numbers*, Amer. J. Math. **128** (2006).
3. Claudiu Raicu, Steven V Sam, Jerzy Weyman, and Fuxiang Yang, *Powers of binary forms and derived Hermite reciprocity*, arXiv **2602.15175** (2026).

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