Commutative Algebra and Algebraic Geometry in Milan

Politecnico di Milano, June 29–July 3, 2026

Hongmiao Yu

On the weak and strong Lefschetz properties for initial ideals of determinantal ideals with respect to diagonal monomial orders

Politecnico di Torino hongmiao.yu@polito.it

The weak and strong Lefschetz properties

Let $S = K[X_1, \dots, X_N]$ be a standard graded polynomial ring over a filed K, I a homogeneous ideal of S and let d be the Krull dimension of S/I. We say that S/I has the **weak Lefschetz property (WLP)** if S/I is Cohen–Macaulay and there exists a linear system of parameters $\underline{\theta} = \theta_1, \dots, \theta_d \in S_1$ of S/I and a linear form $L \in S_1$ such that the multiplication map

$$\times L : (S/(I, \underline{\theta}))_j \longrightarrow (S/(I, \underline{\theta}))_{j+1}$$

has maximal rank for all j, that is, $\times L$ is either injective or surjective; we say that S/I has the **strong Lefschetz property (SLP)** if the multiplication map

$$\times L^s : (S/(I, \underline{\theta}))_j \longrightarrow (S/(I, \underline{\theta}))_{j+s}$$

has maximal rank for all j and for all s.

Lefschetz properties under Gröbner degenerations

Proposition (Wiebe, 2004)

Let $S = K[X_1, \dots, X_N]$ be a standard graded polynomial ring and let I be a homogeneous ideal of S. If $\mathrm{gin}(I)$ is the generic initial ideal of I with respect to the reverse lexicographic order, then S/I has the WLP (resp. SLP) if and only if $S/\mathrm{gin}(I)$ has the WLP (resp. SLP).

Proposition (Wiebe, 2004; Murai, 2010)

Let $I \subseteq S = K[X_1, \dots, X_N]$ be a homogenous ideal and let $\mathrm{in}_<(I)$ be the initial ideal of I with respect to a term order $<$. If $S/\mathrm{in}_<(I)$ has the WLP (resp. SLP), then the same holds for S/I.

However, in general,

S/I has the WLP (resp. SLP) $\not\Rightarrow S/\mathrm{in}_<(I)$ has the WLP (resp. SLP)

Question (Murai, 2018)

*Suppose that $\mathrm{in}_<(I)$ is a **square-free** monomial ideal and that S/I has the WLP (resp. SLP). Does it follow that $S/\mathrm{in}_<(I)$ also has the WLP (resp. SLP)?*

Why "**square-free**"?

Theorem (Conca-Varbaro, 2020)

Let I be a homogeneous ideal of S such that $\mathrm{in}_<(I)$ is a square-free monomial ideal for some term order $<$. Then S/I is Cohen-Macaulay if and only if $S/\mathrm{in}_<(I)$ is Cohen-Macaulay.

Answering Murai's question negatively

Soll and Welker defined a monomial order $\preceq$ and conjectured that

> *Assume that $m = n$. The simplicial complex Δ defined by $\mathrm{in}_\preceq(I_t)$ is a simplicial $(d-1)$-sphere for all $t \leq n$.*

In 2009, Rubey and Stump provided a complete proof of this conjecture[1].

Moreover, a result of Adiprasito states that

> if Δ is a simplicial $(d-1)$-sphere, then $K[\Delta]$ has the SLP.

Consequently, $R/\mathrm{in}_\prec(I_t)$ has the SLP for all $t \leq n = m$. By the previous result of Murai, it follows that R/I_t also has the SLP for all $t \leq n = m$. Therefore, for $m = n \geq t+2$, the Main Theorem provides a family of ideals I_t such that

R/I_t has the SLP and $\mathrm{in}(I_t)$ is square-free, but $R/\mathrm{in}(I_t)$ fails the WLP.

1. However, this proof was not formally published.

Setup and notations

- $X = (X_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n}$ is an $m \times n$ matrix of indeterminates;
- $R = K[X] = K[X_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n]$ is a standard graded polynomial ring over a field of *characteristic zero*;
- for a positive integer $t \in \mathbb{N}$ such that $2 \leq t \leq \min\{m, n\}$ and $t < \max\{m, n\}$, I_t is the ideal of R generated by t-minors of X;
- denote by $\mathrm{in}(I_t)$ the initial ideal of I_t with respect to a *diagonal monomial order*.

Lefschetz properties for $R/\mathrm{in}(I_t)$: Case $t = \min\{m, n\}$

It is known that

> *if $t = \min\{m, n\}$, then $\mathrm{in}(I_t)$ has a t-linear resolution.*

Moreover, a result proved by Cavaliere, Rossi, and Valla in 1990 implies that if S/I is Artinian, the ideal I has a d-linear resolution if and only if $I = \mathfrak{m}^d$. Since $S/\mathfrak{m}^d$ has the SLP for all d, it follows that

Theorem (-)

Let $R = K[X] = K[X_{i,j} \mid 1 \leq i \leq m, 1 \leq j \leq n]$ be a standard graded polynomial ring and $t = \min\{m, n\}$. Then $R/\mathrm{in}(I_t)$ has the SLP.

Lefschetz properties for $R/\mathrm{in}(I_t)$: Case $t < \min\{m, n\}$

Lemma (-)

For $2 \leq t < \min\{m, n\}$, if one of the following conditions holds:

i) $t = 2$ *and* $mn \geq 16$,
ii) $t = 3$ *and* $mn \geq 24$,
iii) $t = 4$ *and* $mn \geq 35$,
iv) $t \geq 5$ *and* $mn \geq (t+1)(t+2)$,

then $\mathrm{HF}(R/(\mathrm{in}(I_t), \underline{\theta}), t-1) \leq \mathrm{HF}(R/(\mathrm{in}(I_t), \underline{\theta}), t)$.

Lemma (-)

For all t such that $2 \leq t \leq \min\{m, n\}$ and $t < \max\{m, n\}$, the graded Betti number

$$\beta_{h,h+t-1}(R/\mathrm{in}(I_t)) \geq t,$$

where $h = (m-t+1)(n-t+1)$ is the height of I_t.

Moreover, a Macaulay2 computation showed that if $t = 4$ and $(m, n) \in \{(5, 6), (6, 5)\}$, then $R/\mathrm{in}(I_4)$ fails the WLP. Therefore,

Main Theorem (-)

Let $2 \leq t < \min\{m, n\}$. Then $R/\mathrm{in}(I_t)$ fails the WLP if one of the following conditions holds:

i) $t = 2$ *and* $mn \geq 16$,
ii) $t = 3$ *and* $mn \geq 24$,
iii) $t \geq 4$ *and* $mn \geq (t+1)(t+2)$.

In particular, if $m = n$, then $R/\mathrm{in}(I_t)$ fails the WLP for all $n \geq t+2$.

On the sharpness of the bound in the Main Theorem

One can verify using Macaulay2 computations that

- if $t = 2$ and $mn \leq 15$, then $R/\mathrm{in}(I_t)$ has the SLP;
- if $t = 3$ and $(m, n) \in \{(4, 5), (5, 4)\}$, then $R/\mathrm{in}(I_3)$ has the WLP;
- for a specific integer $2 \leq t = m - 1 = n - 1$, we have that $R/\mathrm{in}(I_t)$ has the SLP.

Therefore, if the following question has a positive answer:

Does the equality $\mathrm{gin}(\mathrm{in}(I_t)) = \mathrm{gin}(I_t)$ hold for all $t = m - 1 = n - 1$?

then the bound provided in the Main Theorem is sharp.

Sponsored by: MUR PRIN

PRIN 2022 (MUR grant 2022K48YYP)

Unirationality, Hilbert schemes and Singularities

Foundation Compositio Mathematica