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Multigraded Betti Numbers of Veronese Embeddings

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Introduction

A central open question in the study of syzygies is to determine the Betti table of $\mathbb{P}^m$ under the d-uple Veronese embedding. The case $m = 1$ is well understood, but even the case $m = 2$ is wide open. For the $m = 2$ case, in a major computational effort, [1] recently computed all multigraded Betti numbers up to the sixth Veronese (and some up to the eighth).

Our goal is to better understand the multigraded Betti numbers of Veronese embeddings of projective spaces. In particular, we prove new **vanishing and non-vanishing results** for these multigraded Betti numbers.

Background

Let k be any field. Fix integers $d \geq 2$ and $m \geq 2$. We consider the set $\mathcal{A} = \{\mathbf{a} \in \mathbb{N}^{m+1} \mid |\mathbf{a}| = d\}$, where $|\mathbf{a}| := \mathbf{a}_0 + \mathbf{a}_1 + \cdots + \mathbf{a}_m$ denotes the sum of all coordinates of $\mathbf{a}$. Let $\mathbf{a}^i$ be the i-th element of $\mathcal{A}$ with respect to the lexicographical order, $n := \binom{d+m}{m}$ the cardinality of $\mathcal{A}$, and $\mathbb{N}\mathcal{A} \subset \mathbb{N}^{m+1}$ the semigroup generated by $\mathcal{A}$.

Consider the semigroup homomorphism

$$\pi\colon \mathbb{N}^n \to \mathbb{Z}^{m+1}, \quad \mathbf{u} = (u_1, \dots, u_n) \mapsto u_1\mathbf{a}^1 + \cdots + u_n\mathbf{a}^n.$$

Let $S = k[x_1, \dots, x_n]$. The homomorphism π defines a multigrading on S, where $\deg(x_i) = \mathbf{a}^i$. And this induces a k-algebra homomorphism $\hat{\pi}\colon S \to k[\mathbb{N}\mathcal{A}]$. Note that $k[\mathbb{N}\mathcal{A}]$ is the d^{th} Veronese subring of $k[y_0, y_1, \dots, y_m]$, and it has a multigraded S-module structure via the map $\hat{\pi}$.

We consider the minimal multigraded free resolution of $k[\mathbb{N}\mathcal{A}]$.

$$0 \leftarrow k[\mathbb{N}\mathcal{A}] \xleftarrow{\phi_0} \bigoplus_{\mathbf{a}\in\mathbb{Z}^{m+1}} S(-\mathbf{a})^{\beta_{0,\mathbf{a}}} \xleftarrow{\phi_1} \bigoplus_{\mathbf{a}\in\mathbb{Z}^{m+1}} S(-\mathbf{a})^{\beta_{1,\mathbf{a}}} \leftarrow \cdots \leftarrow \bigoplus_{\mathbf{a}\in\mathbb{Z}^{m+1}} S(-\mathbf{a})^{\beta_{n,\mathbf{a}}} \leftarrow 0.$$

The quantities $\beta_{p,\mathbf{a}}$ are called the **multigraded Betti numbers** of $k[\mathbb{N}\mathcal{A}]$.

The multigraded Betti number $\beta_{p,\mathbf{a}}$ measures the minimal number of generators required in degree $\mathbf{a}$ for $\operatorname{im}\phi_p$.

Geometrically, this corresponds to studying the multigraded Betti numbers of the coordinate ring of $\mathbb{P}^m$ under the d-uple Veronese embedding $\mathbb{P}^m \to \mathbb{P}^{n-1}$.

Betti Numbers via Simplicial Complexes

For any degree $\mathbf{b} \in \mathbb{N}\mathcal{A}$, we define the simplicial complex $\Delta_{\mathbf{b}}$ by

$$\Delta_{\mathbf{b}} = \left\{ I \subset \{1, \dots, n\} \mid \mathbf{b} - \sum_{i\in I} \mathbf{a}^i \in \mathbb{N}\mathcal{A} \right\}.$$

Hochster's Formula: $\beta_{p,\mathbf{b}} = \dim_k \tilde{H}_{p-1}(\Delta_{\mathbf{b}}; k)$.

Vanishing Theorem

Let $\mathbf{b} \in \mathbb{N}\mathcal{A}$ with $|\mathbf{b}| = dj$ for a positive integer j. We give lower and upper bounds for the components of $\mathbf{b}$ beyond which $\Delta_{\mathbf{b}}$ is a **cone** and consequently the corresponding Betti numbers vanish.

Theorem 1

If $\mathbf{b}_0 \geq A_j$, then $\beta_{p,\mathbf{b}} = 0$ for all $p \in \mathbb{Z}$.

Here, $A_j = A_j(m, d) := \sum_{i=1}^{j} \mathbf{a}_0^i$ is a combinatorial bound.

Theorem 2

The bound A_j is sharp.

Theorem 3

Let $j \geq d + 1$. If $\mathbf{b}_0 \leq \tilde{l}_j$, then $\beta_{p,\mathbf{b}} = 0$ for all $p \in \mathbb{Z}$.

Again, $\tilde{l}_j = \tilde{l}_j(m, d)$ is a combinatorial bound, but we do not have closed formulae.

Table for the Veronese Surface

For the Veronese surface case, we present tables showing the two bounds for cases $d = 2, 3$, and 4.

$d = 2$

j	3	4	5	$j \geq 6$
$\tilde{l}_j$	0	1	3	3
A_j	4	4	4	4

$d = 3$

j	4	5	6	7	8	9	$j \geq 10$
$\tilde{l}_j$	0	0	1	3	5	8	9
A_j	8	9	10	10	10	10	10

$d = 4$

j	5	6	7	8	9	10	11	12	13	14	$j \geq 15$
$\tilde{l}_j$	0	0	1	2	3	5	7	9	13	17	19
A_j	14	16	17	18	19	20	20	20	20	20	20

Nonvanishing Theorem

Applying **Forman's discrete Morse theory** [2], we compute the exact Betti numbers when $\mathbf{b}_0 = A_{p+1} - 1$ and state the result as follows.

Theorem 4

Let $m \leq p \leq \binom{d+m-1}{m} - 1$. Suppose $|\mathbf{b}| = d(p+1)$ and $\mathbf{b}_0 = A_{p+1} - 1$. We define a set D that depends on p and $\mathbf{b}$. Then the simplicial complex $\Delta_{\mathbf{b}}$ is homotopy equivalent to the wedge sum of $\#D$ copies of the sphere S^{p-1}. Hence, the Betti number $\beta_{p,\mathbf{b}} = \#D$.

Example: The 3-uple Veronese Embedding $\mathbb{P}^2 \to \mathbb{P}^9$

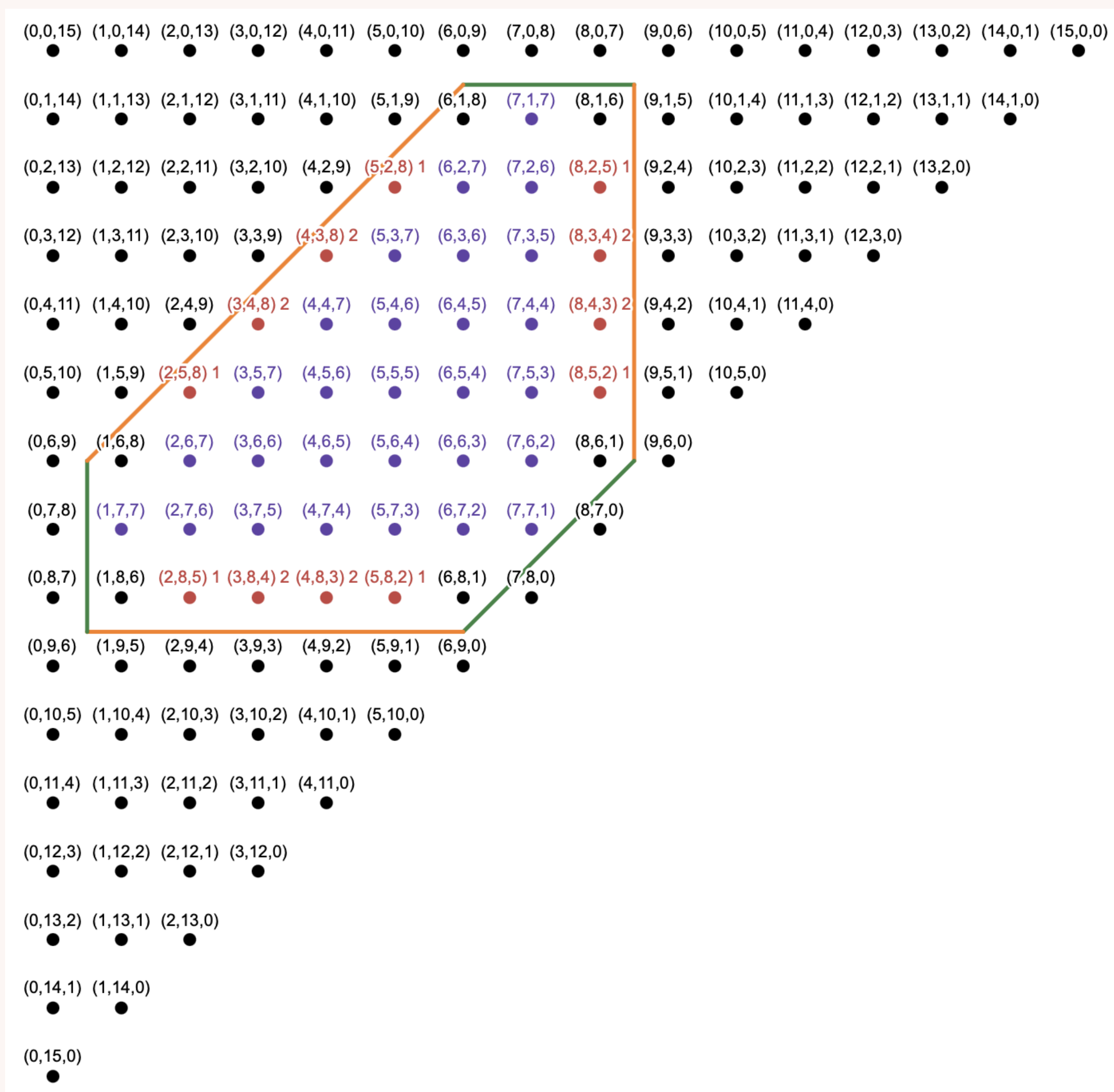

Figure: Multigraded Betti numbers $\beta_{4,\mathbf{b}}$ for $|\mathbf{b}| = 15$.

Bibliography

[1] Juliette Bruce, Daniel Erman, Steve Goldstein, and Jay Yang. Conjectures and computations about Veronese syzygies. Exp. Math., 29(4):398–413, 2020.

[2] Robin Forman. A user's guide to discrete Morse theory. Sémin. Lothar. Comb., 48:b48c, 35, 2002.

[3] Christian Haase and Zongpu Zhang. Multigraded Betti numbers of Veronese embeddings. Preprint, arXiv:2511.18994 [math.AC] (2026).

[4] The full list of references can be found in our preprint arXiv:2511.18994.

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